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Continuity envelopes and sharp embeddings in spaces of generalized smoothness ☆

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Abstract

We study continuity envelopes in spaces of generalized smoothness $B^{\boldsymbol{\sigma},N}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},N}_{p,q}(\mathbb{R}^n)$. The results are applied in proving sharp embedding assertions in some limiting situations.

Keywords: Spaces of generalized smoothness; Limiting embeddings; Continuity envelopes

1. Introduction

This paper continues the study of sharp embeddings and continuity envelopes in spaces of generalized smoothness begun in [17] and [6]. Moreover, as we shall immediately explain, there also appear close connections to [5] and [7] dealing with *growth envelopes* in such spaces, also related to certain '*limiting*' situations. In [17] and [6] we considered spaces of type $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, with $0 < p, q \leqslant \infty$ ($p < \infty$ for F-spaces), $\frac{n}{p} < s \leqslant 1 + \frac{n}{p}$, where Ψ is a so-called admissible function, typically of log-type near 0. Our present paper covers these results essentially.

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The study of spaces of generalized smoothness has a long history, resulting on one hand from the interpolation side (with a function parameter), see [21] and [9], whereas the rather abstract approach (approximation by series of entire analytic functions and coverings) was independently developed by Gol'dman and Kalyabin in the late 1970s and early 1980s of the last century; we refer to the survey [19] and the appendix [20] which cover the extensive (Russian) literature at that time. We rely on the Fourier-analytical approach as presented in [13] recently. There one can also find a reason for the revived interest in the study of such spaces: its connection with applications for pseudo-differential operators (as generators of sub-Markovian semi-groups). Plainly these latter applications and also the topic in its full generality are out of the scope of the present paper. Likewise we only want to mention that the increased interest in function spaces of generalized smoothness is also connected with the study of trace spaces on fractals, that is, so-called h-sets Γ.

Roughly speaking, spaces of generalized smoothness extend 'classical' spaces of Besov or Triebel–Lizorkin type in the sense that the partition of $\mathbb{R}^n$ is not necessarily based on annuli, and, accordingly, the weight factor connected with the assumed smoothness of the distribution, can vary in a wider sense, that is, the 'classical' smoothness $s \in \mathbb{R}$ is replaced by a certain sequence $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$. In the special case of sequences $\boldsymbol{\sigma} = (2^{js})_{j\in\mathbb{N}_0}$, $s \in \mathbb{R}$ and $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$ we have the coincidence $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$, but the new setting is by no means restricted to this case. We benefit in this paper from a result in [24] saying that under certain assumptions on the involved sequences,

$$\big\| f \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \big\|^{(k)} := \| f | L_p \| + \left(\sum_{j=0}^{\infty} \sigma_j^q \, \omega_k\big(f, N_j^{-1}\big)_p^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$, where $k \in \mathbb{N}$ must be sufficiently large.

In contrast to the notion of spaces of generalized smoothness, the study of continuity envelopes has a rather short history; this new tool was developed only recently in [15,16,27], initially intended for a more precise characterization of function spaces. It turned out, however, that it leads not only to surprisingly sharp results based on classical concepts, but allows a lot of applications, too, e.g. to the study of compact embeddings. We return to this point later. Roughly speaking, a continuity envelope $\mathfrak{E}_{\mathsf{C}}(X)$ of a function space X consists of a so-called continuity envelope function

$$\mathcal{E}^X_{\mathsf{C}}(t) \sim \sup_{\|f|X\|\leqslant 1} \frac{\omega(f,t)}{t}, \quad t > 0, \tag{1}$$

together with some '*fine index*' u_X; here $\omega(f,t)$ stands for the modulus of continuity, as usual. Forerunners of *continuity envelopes* in a wider sense are well known for decades, we refer to [27] and [16] for further details and historical comments. In the context of spaces of generalized smoothness see also [18] (in addition to the very recent contributions mentioned above).

Denoting by A either B or F, the main objective is to characterize the above continuity envelopes (1) of spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ when these spaces are not continuously embedded into the Lipschitz space $\mathrm{Lip}^1(\mathbb{R}^n)$; for that reason we shall first prove a criterion for this embedding to hold.

Moreover, when $\boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, $s \in \mathbb{R}$, Ψ an *admissible* function (that is, essentially of log-type, including, in particular, $\Psi \equiv 1$), and $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$, then $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n) = A^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ and we proved in [17] and [6] that

$$\mathcal{E}_{\mathsf{C}}^{A^{(s,\Psi)}_{p,q}}(t) \sim \Phi_{\frac{n}{1-\delta},u'_A}\bigl(t^n\bigr), \quad 0 < t < \frac{1}{2},$$

with $0 < \delta = s - \frac{n}{p} \leqslant 1$ and $u_A = q$ when $A = B$ and $u_A = p$ when $A = F$, $0 < u_A \leqslant \infty$, $\frac{1}{u'_A} = \max(1 - \frac{1}{u_A}, 0)$. The auxiliary function $\Phi_{r,u} : (0, 2^{-n}] \to \mathbb{R}$, $0 < r, u \leqslant \infty$, introduced first in [8], is defined by

$$\Phi_{r,u}(t) := \left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}u}\Psi(y)^{-u}\,\frac{\mathrm{d}y}{y} \right)^{1/u}$$

(suitably modified if $u = \infty$). In this paper we extend this result as follows. For $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ admissible sequences satisfying some further additional conditions, and a certain function Λ with $\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, then

$$\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \sim \left(\int_{t}^{N_0^{-1}} \Lambda\bigl(y^{-1}\bigr)^{-q'} y^{-(\frac{n}{p}+1)q'}\,\frac{\mathrm{d}y}{y} \right)^{1/q'} \tag{2}$$

if $1 < q \leqslant \infty$, otherwise, for $0 < q \leqslant 1$, appropriately modified by

$$\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \sim \sup_{y\in[t,N_0^{-1}]} \Lambda\bigl(y^{-1}\bigr)^{-1} y^{-(\frac{n}{p}+1)}. \tag{3}$$

There is a parallel result for F-spaces. In fact, we can give even more precise characterizations leading to the concept of so-called *continuity envelopes*, where the continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$ is complemented by the fine index u_X. Moreover, we obtain sharp embedding results, namely—as already announced above—that

$$A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\bigl(\mathbb{R}^n\bigr) \hookrightarrow \mathrm{Lip}^1\bigl(\mathbb{R}^n\bigr) \quad \text{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_{u'_A}, \tag{4}$$

using the above abbreviation for u_A again, and assuming for the lower Boyd index $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p}) > 0$. We also prove criteria for the embedding

$$A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1}\bigl(\mathbb{R}^n\bigr) \hookrightarrow A^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2}\bigl(\mathbb{R}^n\bigr) \tag{5}$$

in the limiting case $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}) = \bar{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$, complementing earlier outcomes in [5,7], and extending [6].

Having in mind [17] and [6] one can expect that our envelope results can also be applied to get (sharp) estimates for approximation numbers of embeddings of type $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U) \to A^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,\infty}(U)$, where U stands for the unit ball in $\mathbb{R}^n$, but this will be out of consideration in this paper.

The paper is organized as follows. We collect the necessary background material in Section 2. In Section 3 we obtain criteria for sharp embeddings of type (4). This is not only needed afterwards, but also of some interest of its own. Our main result on continuity envelopes in spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ can be found in Section 4, whereas Section 5 contains some applications: Hardy-type inequalities, and our main sharp embedding result (5).

2. Preliminaries

2.1. General notation

We shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$, $n \in \mathbb{N}$, denotes the n-dimensional real Euclidean space and $\mathbb{R} = \mathbb{R}^1$. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \phi(r) \sim \psi(r)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leqslant b_k \leqslant c_2 a_k \quad \text{or} \quad c_1\,\phi(x) \leqslant \psi(x) \leqslant c_2\,\phi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and ϕ, ψ are non-negative functions. If $a \in \mathbb{R}$ then $a_+ := \max(a, 0)$. If $0 < u \leqslant \infty$, the number u' is given by $1/u' := (1 - 1/u)_+$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. If not otherwise indicated, log is always taken with respect to base 2.

We consider here only function spaces defined on $\mathbb{R}^n$; so for convenience we shall usually omit the "$\mathbb{R}^n$" from their notation.

2.2. Sequences

In this subsection we explain the class of sequences we shall be interested in and some related basic results.

A sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is said to be *admissible* if there exist two positive constants d_0 and d_1 such that

$$d_0\gamma_j \leqslant \gamma_{j+1} \leqslant d_1\gamma_j, \quad j \in \mathbb{N}_0. \tag{6}$$

Clearly, for admissible sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$, $\boldsymbol{\gamma\tau} := (\gamma_j\tau_j)_{j\in\mathbb{N}_0}$ and $\boldsymbol{\gamma}^r := (\gamma_j^r)_{j\in\mathbb{N}_0}$, $r \in \mathbb{R}$, are admissible, too.

For an admissible sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$, let

$$\underline{\gamma}_j := \inf_{k\geqslant 0} \frac{\gamma_{j+k}}{\gamma_k} \quad \text{and} \quad \overline{\gamma}_j := \sup_{k\geqslant 0} \frac{\gamma_{j+k}}{\gamma_k}, \quad j \in \mathbb{N}_0. \tag{7}$$

Then clearly $\underline{\gamma}_j \gamma_k \leqslant \gamma_{j+k} \leqslant \gamma_k \overline{\gamma}_j$, for any $j, k \in \mathbb{N}_0$. In particular, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best possible constants d_0 and d_1 in (6), respectively. The lower and upper Boyd indices of the sequence $\boldsymbol{\gamma}$ are defined, respectively, by

$$\underline{s}(\boldsymbol{\gamma}) := \lim_{j\to\infty} \frac{\log \underline{\gamma}_j}{j} \quad \text{and} \quad \bar{s}(\boldsymbol{\gamma}) := \lim_{j\to\infty} \frac{\log \overline{\gamma}_j}{j}. \tag{8}$$

The above definition is well posed: the sequence $(\log \overline{\gamma}_j)_{j\in\mathbb{N}}$ is sub-additive and hence the right-hand side limit in (8) exists, it is finite (since $\boldsymbol{\gamma}$ is an admissible sequence) and it coincides with $\inf_{j>0} \log \overline{\gamma}_j/j$. The corresponding assertions for the lower counterpart $\underline{s}(\boldsymbol{\gamma})$ can be read off observing that $\log \underline{\gamma}_j = -\log(\overline{\gamma^{-1}})_j$.

Remark 2.1. The Boyd index $\bar{s}(\boldsymbol{\gamma})$ of an admissible sequence $\boldsymbol{\gamma}$ describes the asymptotic behaviour of the $\overline{\gamma}_j$'s and provides more information than simply $\overline{\gamma}_1$ and, what is more, is stable under the equivalence of sequences: if $\boldsymbol{\gamma} \sim \boldsymbol{\tau}$, then $\bar{s}(\boldsymbol{\gamma}) = \bar{s}(\boldsymbol{\tau})$ as one readily verifies. In general, we have for admissible sequences $\boldsymbol{\gamma}, \boldsymbol{\tau}$ that

$$\bar{s}\big(\boldsymbol{\gamma}^r\big) = r\bar{s}(\boldsymbol{\gamma}), \quad r \geqslant 0, \qquad \bar{s}(\boldsymbol{\gamma}\boldsymbol{\tau}) \leqslant \bar{s}(\boldsymbol{\gamma}) + \bar{s}(\boldsymbol{\tau}); \tag{9}$$

similarly for the lower counterpart. Observe that, given $\varepsilon > 0$, there are two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1 2^{(\underline{s}(\boldsymbol{\gamma})-\varepsilon)j} \leqslant \underline{\gamma}_j \leqslant \overline{\gamma}_j \leqslant c_2 2^{(\bar{s}(\boldsymbol{\gamma})+\varepsilon)j}, \quad j \in \mathbb{N}_0. \tag{10}$$

From (10) it follows that for $\underline{s}(\boldsymbol{\gamma}) > 0$, then $\boldsymbol{\gamma}^{-1} \in \ell_u$ for arbitrary $u \in (0, \infty]$. Conversely, $\bar{s}(\boldsymbol{\gamma}) < 0$ implies $\boldsymbol{\gamma}^{-1} \notin \ell_\infty$, that is, $\boldsymbol{\gamma}^{-1}$ does not belong to any ℓ_u, $0 < u \leqslant \infty$.

Examples 2.2. We consider some examples of admissible sequences.

(i) The sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$,

$$\gamma_j = 2^{sj}(1+j)^b\big(1+\log(1+j)\big)^c$$

with arbitrary fixed real numbers s, b and c is a standard example of an admissible sequence with $\underline{s}(\boldsymbol{\gamma}) = \bar{s}(\boldsymbol{\gamma}) = s$.

(ii) Let $\Phi : (0,1] \to \mathbb{R}$ be a slowly varying function (or equivalent to a slowly varying one) in the sense of [2]. Then, for $s \in \mathbb{R}$ the sequence $\boldsymbol{\gamma} = (2^{sj}\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ is an admissible sequence. Also here we have $\underline{s}(\boldsymbol{\gamma}) = \bar{s}(\boldsymbol{\gamma}) = s$.

(iii) In view of [3, Proposition 1.9.7], the case $\boldsymbol{\gamma} = (2^{sj}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, where now Ψ is an admissible function in the sense of [11] (i.e., a positive monotone function defined on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}_0$), can be regarded as a special case of (ii).

Remark 2.3. The examples above have in common the fact that their upper and lower Boyd indices coincide. However, this is not in general the case. In [13] one can find examples, due to Kalyabin, showing that an admissible sequence has not necessarily a fixed main order. But one can even say more than this. As a consequence of [4, Proposition 7.3], for any $0 < a \leqslant b < \infty$, there is an admissible sequence $\boldsymbol{\gamma}$ with $\underline{s}(\boldsymbol{\gamma}) = a$ and $\bar{s}(\boldsymbol{\gamma}) = b$, that is, with prescribed upper and lower Boyd indices.

2.3. Function spaces of generalized smoothness

Let $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ (recall (7)). In particular $\boldsymbol{N}$ is a so-called strongly increasing sequence (cf. [13, Definition 2.2.1]) which guarantees the existence of a number $l_0 \in \mathbb{N}_0$ such that

$$N_k \geqslant 2N_j \quad \text{for any } k, j \text{ such that } k \geqslant j + l_0. \tag{11}$$

It should be noted that the sequence $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ plays the same rôle as the sequence $(2^j)_{j\in\mathbb{N}_0}$ in the classical construction of the spaces B^s_{pq} and F^s_{pq}. This will be clear from the following considerations.

For a fixed sequence $\boldsymbol{N}$ as above we define the associated covering $\Omega^{\boldsymbol{N}} = (\Omega^{\boldsymbol{N}}_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega^{\boldsymbol{N}}_j = \bigl\{\xi \in \mathbb{R}^n \colon |\xi| \leqslant N_{j+l_0}\bigr\}, \quad j = 0, \dots, l_0 - 1,$$

and

$$\Omega^{\boldsymbol{N}}_j = \bigl\{\xi \in \mathbb{R}^n \colon N_{j-l_0} \leqslant |\xi| \leqslant N_{j+l_0}\bigr\}, \quad j \geqslant l_0,$$

with l_0 according to (11).

Definition 2.4. For a fixed admissible sequence $\boldsymbol{N}$ with $\underline{N}_1 > 1$, and for the associated covering $\Omega^{\boldsymbol{N}} = (\Omega^{\boldsymbol{N}}_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$, a system $\varphi^{\boldsymbol{N}} = (\varphi^{\boldsymbol{N}}_j)_{j\in\mathbb{N}_0}$ will be called a (generalized) partition of unity subordinate to $\Omega^{\boldsymbol{N}}$ if:

(i) $\varphi^{\boldsymbol{N}}_j \in C^\infty_0$ and $\varphi^{\boldsymbol{N}}_j(\xi) \geqslant 0$ if $\xi \in \mathbb{R}^n$ for any $j \in \mathbb{N}_0$;
(ii) $\operatorname{supp}\varphi^{\boldsymbol{N}}_j \subset \Omega^{\boldsymbol{N}}_j$ for any $j \in \mathbb{N}_0$;
(iii) for any $\alpha \in \mathbb{N}^n_0$ there exists a constant $c_\alpha > 0$ such that for any $j \in \mathbb{N}_0$

$$\bigl|\mathrm{D}^\alpha \varphi^{\boldsymbol{N}}_j(\xi)\bigr| \leqslant c_\alpha \bigl(1 + |\xi|^2\bigr)^{-|\alpha|/2} \quad \text{for any } \xi \in \mathbb{R}^n;$$

(iv) there exists a constant $c_\varphi > 0$ such that

$$0 < \sum_{j=0}^{\infty} \varphi^{\boldsymbol{N}}_j(\xi) = c_\varphi < \infty \quad \text{for any } \xi \in \mathbb{R}^n.$$

Before turning to the definition of the spaces of generalized smoothness let us recall that $\mathcal{S}$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology and by $\mathcal{S}'$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. For $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$ we will use the notation $\varphi(\mathrm{D})f = [\mathcal{F}^{-1}(\varphi\mathcal{F}f)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand, respectively, for the Fourier and inverse Fourier transform. Furthermore, if $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ then L_p and ℓ_q have the standard meaning and, if $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\left\|(f_j)_{j\in\mathbb{N}_0}\middle|\ell_q(L_p)\right\| := \left(\sum_{j=0}^{\infty}\|f_j|L_p\|^q\right)^{1/q}$$

and

$$\left\|(f_j)_{j\in\mathbb{N}_0}\middle|L_p(\ell_q)\right\| := \left\|\left(\sum_{j=0}^{\infty}\left|f_j(\cdot)\right|^q\right)^{1/q}\middle|L_p\right\|$$

with the appropriate modification if $q = \infty$.

Definition 2.5. Let $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ and $\varphi^{\boldsymbol{N}}$ be a system of functions as in Definition 2.4. Let $0 < q \leqslant \infty$ and $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence.

(i) Let $0 < p \leqslant \infty$. The Besov space of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ is the set of all tempered distributions f such that the quasi-norm

$$\left\|f\middle|B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right\| := \left\|\left(\sigma_j\varphi_j^{\boldsymbol{N}}(\mathrm{D})f\right)_{j\in\mathbb{N}_0}\middle|\ell_q(L_p)\right\|$$

is finite.

(ii) Let $0 < p < \infty$. The Triebel–Lizorkin space of generalized smoothness $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ is the set of all tempered distributions f such that the quasi-norm

$$\left\|f\middle|F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right\| := \left\|\left(\sigma_j\varphi_j^{\boldsymbol{N}}(\mathrm{D})f(\cdot)\right)_{j\in\mathbb{N}_0}\middle|L_p(\ell_q)\right\|$$

is finite.

Remark 2.6. Note that if $1 < p < \infty$ one can consider a weaker assumption on the sequence $\boldsymbol{N}$ in the above definition. We refer to [13] for a discussion on this, as well as for some historical references to the subject and a systematic study of these spaces, including a characterization by local means and atomic decomposition.

Remark 2.7. If $\boldsymbol{\sigma} = (2^{sj})_{j\in\mathbb{N}_0}$, with s a real number, and $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$ then the spaces $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ coincide with the usual Besov or Triebel–Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$, respectively. If we let $\boldsymbol{\sigma} = (2^{sj}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, where Ψ is an admissible function in the sense of [11,12] (see Example 2.2(iii)), the corresponding Besov space coincides with the space $B^{(s,\Psi)}_{p,q}$ introduced by Edmunds and Triebel in [11,12] and also considered by Moura in [22,23]. Similarly for the F-counterpart.

Next we recall the definition of differences of functions. If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$, then

$$\left(\Delta_h^k f\right)(x) := \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f(x+jh), \quad x \in \mathbb{R}^n.$$

Note that Δ_h^k can also be defined iteratively via

$$\left(\Delta_h^1 f\right)(x) = f(x+h) - f(x) \quad \text{and} \quad \left(\Delta_h^{k+1} f\right)(x) = \Delta_h^1\left(\Delta_h^k f\right)(x), \quad k \in \mathbb{N}.$$

For convenience we may write Δ_h instead of Δ_h^1. Furthermore, the *kth modulus of smoothness* of a function $f \in L_p$, $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p = \sup_{|h| \leqslant t} \left\| \Delta_h^k f \middle| L_p \right\|, \quad t > 0. \tag{12}$$

We shall simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$ and $\omega(f,t)$ instead of $\omega(f,t)_\infty$.

We recall Marchaud's inequality: let $f \in L_p$, $1 \leqslant p \leqslant \infty$, and $k, r \in \mathbb{N}$ with $k > r$; then, for all $t > 0$,

$$\omega_r(f,t)_p \leqslant c t^r \int_t^\infty \frac{\omega_k(f,u)_p}{u^r} \frac{\mathrm{d}u}{u}, \tag{13}$$

where c is a positive constant which depends only on k and r, see [1, Chapter 5, Eq. (4.7)].

The following result gives a characterization of the Besov spaces of generalized smoothness by means of the modulus of smoothness. We refer to [24].

Theorem 2.8. *Let $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma} = (\sigma_j)_{j \in \mathbb{N}_0}$ and $\boldsymbol{N} = (N_j)_{j \in \mathbb{N}_0}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$, such that $\underline{s}(\boldsymbol{\sigma})\bar{s}(\boldsymbol{N})^{-1} > n(1/p - 1)_+$. Let $k \in \mathbb{N}$ with $k > \bar{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{N})^{-1}$. Then*

$$\left\| f \middle| B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \right\|^{(k)} := \| f | L_p \| + \left(\sum_{j=0}^\infty \sigma_j^q \omega_k\left(f, N_j^{-1}\right)_p^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$.

Remark 2.9. For any $a \in (0, \infty)$,

$$\left\| f \middle| B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \right\|^{(k)} \sim \| f | L_p \| + \left(\int_0^a \left(\Lambda\left(t^{-1}\right) \omega_k(f,t)_p\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

where $\Lambda : (0, \infty) \to (0, \infty)$ is an admissible function—according to [5, Definition 2.2]—such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$.

2.4. Continuity envelopes

The concept of continuity envelopes has been introduced by Haroske in [15] and Triebel in [27]. Here we quote the basic definitions and results concerning continuity envelopes. However, we shall be rather concise and we mainly refer to [16,27] for heuristics, motivations and details on this subject.

Let C be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. Recall that the classical Lipschitz space Lip^1 is defined as the space of all functions $f \in C$ such that

$$\big\| f \big| \mathrm{Lip}^1 \big\| := \|f|C\| + \sup_{t \in (0,1)} \frac{\omega(f,t)}{t} \tag{14}$$

is finite.

Definition 2.10. Let $X \hookrightarrow C$ be some function space on $\mathbb{R}^n$.

(i) The *continuity envelope function* $\mathcal{E}_{\mathsf{C}}^X : (0,\infty) \to [0,\infty)$ is defined by

$$\mathcal{E}_{\mathsf{C}}^X(t) := \sup_{\|f|X\| \leqslant 1} \frac{\omega(f,t)}{t}, \quad t > 0.$$

(ii) Assume $X \not\hookrightarrow \mathrm{Lip}^1$. Let $\varepsilon \in (0,1)$, $H(t) := -\log \mathcal{E}_{\mathsf{C}}^X(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u_X, $0 < u_X \leqslant \infty$, is defined as the infimum of all numbers v, $0 < v \leqslant \infty$, such that

$$\left(\int_0^\varepsilon \left(\frac{\omega(f,t)}{t\mathcal{E}_{\mathsf{C}}^X(t)} \right)^v \mu_H(\mathrm{d}t) \right)^{1/v} \leqslant c\|f|X\| \tag{15}$$

(with the usual modification if $v = \infty$) holds for some $c > 0$ and all $f \in X$. The couple

$$\mathfrak{E}_{\mathsf{C}}(X) = \big(\mathcal{E}_{\mathsf{C}}^X(\cdot), u_X\big)$$

is called *continuity envelope* for the function space X.

As it will be useful in the sequel, we recall some properties of the continuity envelopes. In view of (i) we obtain—strictly speaking—equivalence classes of continuity envelope functions when working with equivalent quasi-norms in X as we shall do in the sequel. However, for convenience we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Note that $\mathcal{E}_{\mathsf{C}}^X$ is equivalent to some monotonically decreasing function; for a proof and further properties we refer to [16]. Concerning (ii) we shall assume that we can choose a continuous representative in the equivalence class $[\mathcal{E}_{\mathsf{C}}^X]$, for convenience (but in a slight abuse of notation) denoted by $\mathcal{E}_{\mathsf{C}}^X$ again. It is obvious that (15) holds for $v = \infty$ and any X. Moreover, one verifies that

$$\sup_{0<t\leqslant\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \leqslant c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{1/v_1} \leqslant c_2 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{1/v_0} \tag{16}$$

for $0 < v_0 < v_1 < \infty$ and all non-negative monotonically decreasing functions g on $(0,\varepsilon]$; cf. [27, Proposition 12.2]. So—passing to a monotonically decreasing function equivalent to $\frac{\omega(f,t)}{t}$, see [10, Chapter 2, Lemma 6.1]—we observe that the left-hand sides in (15) are monotonically ordered in v and it is natural to look for the smallest possible one.

Proposition 2.11.

(i) *Let $X_i \hookrightarrow C$, $i=1,2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t>0$,*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \leqslant c\mathcal{E}_{\mathsf{C}}^{X_2}(t).$$

(ii) *We have $X \hookrightarrow \mathrm{Lip}^1$ if, and only if, $\mathcal{E}_{\mathsf{C}}^{X}$ is bounded.*

(iii) *Let $X_i \hookrightarrow C$, $i=1,2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their continuity envelope functions*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \sim \mathcal{E}_{\mathsf{C}}^{X_2}(t), \quad t \in (0,\varepsilon),$$

for some $\varepsilon>0$. Then we get for the corresponding indices u_{X_i}, $i=1,2$, that

$$u_{X_1} \leqslant u_{X_2}.$$

3. Embeddings

In what follows we present some embedding results which will be useful when discussing continuity envelopes for spaces of generalized smoothness.

We refer to [7, Lemma 4.1] for the following.

Proposition 3.1. *Let $0<p_1<p<p_2\leqslant\infty$, $0<q\leqslant\infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1>1$. Let $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ be the (clearly admissible) sequences defined, respectively, by*

$$\boldsymbol{\sigma}' := \boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p})}\boldsymbol{\sigma} \quad \textit{and} \quad \boldsymbol{\sigma}'' := \boldsymbol{N}^{n(\frac{1}{p_2}-\frac{1}{p})}\boldsymbol{\sigma}.$$

Then

$$B^{\boldsymbol{\sigma}',\boldsymbol{N}}_{p_1,u} \hookrightarrow F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\sigma}'',\boldsymbol{N}}_{p_2,v}$$

if, and only if, $0<u\leqslant p\leqslant v\leqslant\infty$.

Remark 3.2. In case of B-spaces only, one finds in [5, Theorem 3.7] the sharper assertion that for $0<p_1\leqslant p_2\leqslant\infty$, $0<q_1,q_2\leqslant\infty$, admissible sequences $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ and $\boldsymbol{N}$, the latter satisfying $\underline{N}_1>1$, then

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2} \tag{17}$$

if

$$\left(\sigma_j^{-1}\tau_j N_j^{n(\frac{1}{p_1}-\frac{1}{p_2})}\right)_{j\in\mathbb{N}_0} \in \ell_{q^*}, \quad \text{with } \frac{1}{q^*} := \left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+. \tag{18}$$

We return to this observation in Section 5.

For the embedding of the spaces under consideration into C, the space of complex-valued, bounded and uniformly continuous functions on $\mathbb{R}^n$ endowed with the sup-norm, it is known the following.

Proposition 3.3. *Let $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$.*

(i) *Then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_{q'}.$$

(ii) *If $p < \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_{p'}.$$

One can replace C by L_∞ in the assertions above.

We refer to [5, Corollary 3.10 & Remark 3.11] and to [7, Proposition 4.4] concerning parts (i) and (ii), respectively, of the proposition above.

Regarding the embeddings into the classical Lipschitz space we have the following.

Proposition 3.4. *Let $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. We assume that*

$$\underline{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p}\big) > 0. \tag{19}$$

(i) *Then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_{q'}.$$

(ii) *If $p < \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_{p'}.$$

Proof. We just prove here the sufficiency, as the necessity follows from our later results—cf. Remark 4.2, and we start with the proof of (i). Let

$$\boldsymbol{\tau} := \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}. \tag{20}$$

Then, $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}$, as a consequence of Proposition 3.1 and Remark 3.2. By (19), $\underline{s}(\boldsymbol{\tau}) > 0$, so that Theorem 2.8 gives a characterization of $B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}$ by means of differences of order $k \in \mathbb{N}$ for $k > \bar{s}(\boldsymbol{\tau})\underline{s}(\boldsymbol{N})^{-1}$.

Let first $1 < q < \infty$. For any $f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and $0 < t < N_0^{-1}$ we conclude, using also Marchaud's inequality (13), and Hölder's inequality that

$$\begin{aligned}
\frac{\omega(f,t)}{t} &\leqslant c_1 \int_t^{N_0^{-1}} \frac{\omega_k(f,u)}{u}\frac{\mathrm{d}u}{u} + c_2\|f|C\| \\
&\leqslant c_1\left(\int_t^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)\omega_k(f,u)\right)^q \frac{\mathrm{d}u}{u}\right)^{1/q} \\
&\quad\times\left(\int_t^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u}\right)^{1/q'} + c_2\|f|C\| \\
&\leqslant c_3\left\|f\middle|B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}\right\|\left(\left(\int_t^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u}\right)^{1/q'} + 1\right) \\
&\leqslant c_4\left\|f\middle|B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right\|\left(\left(\int_t^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u}\right)^{1/q'} + 1\right),
\end{aligned} \tag{21}$$

where Υ is an admissible function with $\Upsilon(z) \sim \Upsilon(N_j) = \tau_j = \sigma_j N_j^{-n/p}$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. Since

$$\int_0^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u} = \sum_{j=0}^{\infty} \int_{N_{j+1}^{-1}}^{N_j^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u} \sim \sum_{j=0}^{\infty} \left(\sigma_j^{-1} N_j^{n/p+1}\right)^{q'},$$

under the assumption $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_{q'}$, (14) and (21) imply $f \in \mathrm{Lip}^1$ with

$$\left\|f\middle|\mathrm{Lip}^1\right\| \leqslant c\left\|f\middle|B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right\| \tag{22}$$

for $1 < q < \infty$. The case $q = \infty$ (therefore $q' = 1$) can be dealt analogously, with the obvious modifications. When $0 < q \leqslant 1$, so $q' = \infty$, we have

$$\begin{aligned}
\frac{\omega(f,t)}{t} &\leqslant c_1 \int_t^{N_0^{-1}} \Upsilon\left(u^{-1}\right)\omega_k(f,u)\frac{\Upsilon(u^{-1})^{-1}}{u}\frac{\mathrm{d}u}{u} + c_2\|f|C\| \\
&\leqslant c_1 \sup_{u\in[t,N_0^{-1}]} \left(\Upsilon\left(u^{-1}\right)u\right)^{-1} \int_t^{N_0^{-1}} \Upsilon\left(u^{-1}\right)\omega_k(f,u)\frac{\mathrm{d}u}{u} + c_2\|f|C\| \\
&\leqslant c_3\left\|f\middle|B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,1}\right\|\left(\sup_{u\in[t,N_0^{-1}]} \left(\Upsilon\left(u^{-1}\right)u\right)^{-1} + 1\right)
\end{aligned}$$

$$\leqslant c_4 \| f | B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q} \| \Big(\sup_{u\in[t,N_0^{-1}]} \big(\Upsilon(u^{-1})u\big)^{-1} + 1\Big)$$

$$\leqslant c_5 \| f | B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \| \Big(\sup_{u\in[t,N_0^{-1}]} \big(\Upsilon(u^{-1})u\big)^{-1} + 1\Big), \tag{23}$$

for any $f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and $t \in (0, N_0^{-1})$, leading, under the assumption $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_\infty$ (which is equivalent to $\sup_{u\in(0,N_0^{-1})}(\Upsilon(u^{-1})u)^{-1}$ being bounded), to (22) also in this case, which completes the proof of the sufficiency (i).

The sufficiency of (ii) immediately follows by the embedding

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,p},$$

with $\boldsymbol{\tau}$ as in (20)—which is a consequence of Proposition 3.1,—and what has been proved above regarding part (i). □

Remark 3.5. In the particular cases of $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$ and $\boldsymbol{\sigma} = (2^{sj})_{j\in\mathbb{N}_0}$ or $\boldsymbol{\sigma} = (2^{sj}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, according to Remark 2.7, the proposition above is covered by [6, Proposition 2.2, Remark 1.5].

Remark 3.6. Note that

$$\underline{s}(\boldsymbol{\sigma})\bar{s}(\boldsymbol{N})^{-1} > \frac{n}{p} \tag{24}$$

is a sufficient condition for $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p}) > 0$. Moreover, as $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ is increasing, $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \in \ell_{q'}$ implies $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_{q'}$ and thus, by Proposition 3.3, $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C$, as it should be. Similarly for the F-case.

4. Continuity envelopes for $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$

Regarding the study of continuity envelopes in the context of the spaces of generalized smoothness $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$, of interest are those spaces with

$$A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C \quad \text{but} \quad A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \not\hookrightarrow \mathrm{Lip}^1 .$$

Taking into consideration Propositions 3.3 and 3.4 we shall be concerned with the investigation of the continuity envelopes of the spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ with

$$\underline{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p}\big) > 0 \quad \text{and} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p+1} \notin \ell_{u'}, \tag{25}$$

where $u = q$ if $A = B$ and $u = p$ if $A = F$. By Remark 2.1, $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p}) > 0$ implies $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_v$ for any $v \in (0,\infty]$, leading to $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C$ by Proposition 3.3.

Proposition 4.1. *Let $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Assume further that* (25) *holds. Let Λ be any admissible function such that $\Lambda(z) \sim \sigma_j$,*

$z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, *with equivalence constants independent of* j, *and let* $\phi_u^{\boldsymbol{\sigma}}$ *be defined in* $(0, N_0^{-1}]$ *by*

$$\phi_u^{\boldsymbol{\sigma}}(t) := \left(\int_t^{N_0^{-1}} \Lambda\left(y^{-1}\right)^{-u} y^{-(\frac{n}{p}+1)u} \frac{\mathrm{d}y}{y} \right)^{1/u} \quad \text{if } 0 < u < \infty, \tag{26}$$

and

$$\phi_u^{\boldsymbol{\sigma}}(t) := \sup_{y \in [t, N_0^{-1}]} \Lambda\left(y^{-1}\right)^{-1} y^{-(\frac{n}{p}+1)} \quad \text{if } u = \infty. \tag{27}$$

(i) *Then*

$$\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},N}}(t) \sim \phi_{q'}^{\boldsymbol{\sigma}}(t), \quad t \in \left(0, N_1^{-1}\right].$$

(ii) *If* $p < \infty$, *then*

$$\mathcal{E}_{\mathsf{C}}^{F_{p,q}^{\boldsymbol{\sigma},N}}(t) \sim \phi_{p'}^{\boldsymbol{\sigma}}(t), \quad t \in \left(0, N_1^{-1}\right].$$

Proof.

Step 1. From (21) and (23) we get, for any $t \in (0, N_0^{-1})$,

$$\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},N}}(t) \leqslant c \left(\int_t^{N_0^{-1}} \left(\Upsilon\left(u^{-1}\right)u\right)^{-q'} \frac{\mathrm{d}u}{u} \right)^{1/q'} \tag{28}$$

(with the usual modification if $q' = \infty$, i.e., if $0 < q \leqslant 1$), with Υ an admissible function satisfying $\Upsilon(z) \sim \sigma_j N_j^{-n/p}$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. Here we used the first assumption in (25). Clearly, any admissible function Λ with $\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, verifies

$$\Upsilon(z) \sim \sigma_j N_j^{-n/p} \sim \Lambda(z) z^{-n/p}, \quad z \in [N_j, N_{j+1}], \; j \in \mathbb{N}_0$$

(with constants independent of j) which, in (28), immediately leads to

$$\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},N}}(t) \leqslant c\phi_{q'}^{\boldsymbol{\sigma}}(t), \quad t \in \left(0, N_0^{-1}\right).$$

Step 2. In this step we shall prove that, for some positive constant c,

$$\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},N}}(t) \geqslant c\phi_{q'}^{\boldsymbol{\sigma}}(t), \quad t \in \left(0, N_1^{-1}\right]. \tag{29}$$

We take advantage of what has been done in [5], where the so-called local growth envelope for $B_{p,q}^{\boldsymbol{\sigma},N}$ has been studied. In the considerations below we will follow the proof of

Lemma 4.6 and Proposition 4.7 of [5] with the appropriate modifications to our purpose; see also [27, pp. 220–221].

Consider

$$h(y) := e^{-1/(1-y^2)} \quad \text{if } |y| < 1 \quad \text{and} \quad h(y) := 0 \quad \text{if } |y| \geqslant 1.$$

Given $L \in \mathbb{N}_0$ and $\delta \in (0, 1]$, define

$$h_{\delta,L}(y) := h(y) - \sum_{l=0}^{L} \rho_{\delta,\ell} h^{(l)}\big(\delta^{-1}(y - 1 - \delta)\big),$$

where the coefficients $\rho_{\delta,\ell}$ are uniquely determined by imposing that $h_{\delta,L}$ shall obey the following set of conditions:

$$\int_{\mathbb{R}} y^k h_{\delta,L}(y)\,\mathrm{d}y = 0, \quad k = 0, \ldots, L.$$

Let now

$$h^{\delta,L}(y) := \int_0^y h_{\delta,L}(z)\,\mathrm{d}z$$

and

$$\phi^{\delta,L}(x) := h^{\delta,L}(x_1) \prod_{m=2}^{n} h(x_m), \quad x = (x_j)_{j=1}^n \in \mathbb{R}^n. \tag{30}$$

For a fixed $L \in \mathbb{N}_0$ with

$$L > -1 + n\left(\frac{\log_2 \overline{N}_1}{\log_2 \underline{N}_1} \frac{1}{\min(1, p)} - 1\right) - \frac{\log_2 \underline{\sigma}_1}{\log_2 \underline{N}_1},$$

and $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ a sequence of non-negative numbers in ℓ_q, let $f^{\boldsymbol{b}}$ be given by

$$f^{\boldsymbol{b}}(x) := \sum_{j=1}^{\infty} b_j \sigma_j^{-1} N_j^{n/p} \phi^{\delta,L}(N_j x), \quad x \in \mathbb{R}^n. \tag{31}$$

Since the functions

$$a_j(x) := \sigma_j^{-1} N_j^{n/p} \phi^{\delta,L}(N_j x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N},$$

are (up to constants, independently of j) $(\boldsymbol{\sigma}, p)_{M,L} - \boldsymbol{N}$-atoms, for some fixed $M \in \mathbb{N}$ with $M > \log_2 \overline{\sigma}_1 / \log_2 \underline{N}_1$, the atomic decomposition theorem for $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ (cf. [13, Theorem 4.4.3]) yields that $f^{\boldsymbol{b}} \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and

$$\big\| f^{\boldsymbol{b}} \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \big\| \leqslant c \|\boldsymbol{b}|\ell_q\|, \tag{32}$$

for some positive constant c (independent of $\boldsymbol{b}$). Moreover, for a fixed $\eta \in (0,1)$ and $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
& f^{\boldsymbol{b}}(0,0,\ldots,0) - f^{\boldsymbol{b}}(-\eta N_k^{-1},0,\ldots,0) \\
&\quad = c_1 \sum_{j=1}^{\infty} b_j \sigma_j^{-1} N_j^{n/p} \big(h^{\delta,L}(0) - h^{\delta,L}\big(-\eta N_j N_k^{-1}\big)\big) \\
&\quad = c_2 \sum_{j=1}^{\infty} b_j \sigma_j^{-1} N_j^{n/p+1} N_k^{-1} h_{\delta,L}(z_{j,k}) \\
&\quad = c_2 \sum_{j=1}^{\infty} b_j \sigma_j^{-1} N_j^{n/p+1} N_k^{-1} h(z_{j,k}) \\
&\quad \geqslant c_2 \sum_{j=1}^{k} b_j \sigma_j^{-1} N_j^{n/p+1} N_k^{-1} h(z_{j,k}),
\end{aligned}$$

for some $z_{j,k} \in (-\eta N_j N_k^{-1}, 0)$. For $j \leqslant k$, $z_{j,k} \in [-\eta, 0)$, and hence $h(z_{j,k}) \geqslant c > 0$ for some c which is independent of j and k, leading to

$$f^{\boldsymbol{b}}(0,0,\ldots,0) - f^{\boldsymbol{b}}(-\eta N_k^{-1},0,\ldots,0) \geqslant c \sum_{j=1}^{k} b_j \sigma_j^{-1} N_j^{n/p+1} N_k^{-1}.$$

Therefore,

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \geqslant c \sum_{j=1}^{k} b_j \sigma_j^{-1} N_j^{n/p+1}, \quad k \in \mathbb{N}_0. \tag{33}$$

We now take advantage of some choices made in the proof of [5, Proposition 4.8]. Let $1 < q \leqslant \infty$. For each $J \in \mathbb{N}$ denote by f_J the function $f^{\boldsymbol{b}}$ in (31) with $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ being the sequence defined by

$$b_j := \begin{cases} \sigma_j^{1-q'} N_j^{-(\frac{n}{p}+1)(1-q')} \big(\sum_{k=1}^{J} \sigma_k^{-q'} N_k^{(\frac{n}{p}+1)q'}\big)^{-1/q} & \text{for } j = 1,\ldots,J, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $b_j \geqslant 0$, $j \in \mathbb{N}$, $\boldsymbol{b} \in \ell_q$ and, moreover, $\|\boldsymbol{b}|\ell_q\| = 1$, so that (32) and (33) allow us to write, with constants independent of J,

$$\mathcal{E}_{\mathsf{C}}^{B^{\sigma,N}_{p,q}}\big(N_J^{-1}\big) \geqslant \frac{\omega(c_1 f_J, N_J^{-1})}{N_J^{-1}} \geqslant c_2 \left(\sum_{j=1}^{J} \sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'} \right)^{1/q'}, \quad J \in \mathbb{N}. \tag{34}$$

Now let, for each given $t \in (0, N_1^{-1}]$, $J \geqslant 1$ be such that $N_{J+1}^{-1} < t \leqslant N_J^{-1}$. Using the properties of $\mathcal{E}_{\mathsf{C}}^X$, (34) and the hypotheses on $\boldsymbol{\sigma}$, $\boldsymbol{N}$ and Λ, we can derive that

$$\begin{aligned}\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}}(t) &\geqslant c_1 \mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}}\big(N_J^{-1}\big) \geqslant c_2 \Bigg(\sum_{j=1}^{J} \sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'} \Bigg)^{1/q'} \\ &\geqslant c_3 \Bigg(\sum_{j=1}^{J+1} \sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'} \Bigg)^{1/q'} \\ &\geqslant c_4 \Bigg(\int_t^{N_0^{-1}} \Lambda\big(u^{-1}\big)^{-q'} u^{-(\frac{n}{p}+1)q'} \frac{\mathrm{d}u}{u} \Bigg)^{1/q'},\end{aligned}$$

that is (29) for $q > 1$.

If $0 < q \leqslant 1$, for each $J \in \mathbb{N}$ denote by f_J the function $f^{\boldsymbol{b}}$ in (31) with $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ being the sequence defined by

$$b_j := \begin{cases} 1 & \text{for } j = J, \\ 0 & \text{otherwise.} \end{cases}$$

Once again we have $b_j \geqslant 0$, $j \in \mathbb{N}$, $\boldsymbol{b} \in \ell_q$ and, moreover, $\|\boldsymbol{b}|\ell_q\| = 1$. By (33),

$$\frac{\omega(f_J, N_k^{-1})}{N_k^{-1}} \geqslant c\sigma_J^{-1} N_J^{n/p+1}, \quad \text{for any } k \geqslant J,$$

and hence

$$\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}}\big(N_k^{-1}\big) \geqslant c \sup_{j=1,\ldots,k} \sigma_j^{-1} N_j^{(\frac{n}{p}+1)}, \quad k \in \mathbb{N}. \tag{35}$$

So we finally have, for $t \in (0, N_1^{-1}]$, by means of choosing $J \in \mathbb{N}$ such that $N_{J+1}^{-1} < t \leqslant N_J^{-1}$ and with the help of (35), the properties of $\mathcal{E}_{\mathsf{C}}^X$ and the hypotheses on $\boldsymbol{\sigma}$, $\boldsymbol{N}$ and Λ,

$$\begin{aligned}\mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}}(t) &\geqslant c_1 \mathcal{E}_{\mathsf{C}}^{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}}\big(N_J^{-1}\big) \geqslant c_2 \sup_{j=1,\ldots,J} \sigma_j^{-1} N_j^{(\frac{n}{p}+1)} \\ &\geqslant c_3 \sup_{j=1,\ldots,J+1} \sigma_j^{-1} N_j^{(\frac{n}{p}+1)} \\ &\geqslant c_4 \sup_{u\in[t,N_0^{-1}]} \Lambda\big(u^{-1}\big)^{-1} u^{-(\frac{n}{p}+1)},\end{aligned}$$

thus completing the proof of part (i).

Step 3. In this step we shall prove (ii). Let p_1, p_2 be such that $0 < p_1 < p < p_2 \leqslant \infty$ and $\boldsymbol{\sigma}', \boldsymbol{\sigma}''$ be the admissible sequences defined, respectively, by

$$\boldsymbol{\sigma}' := \boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p})}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\sigma}'' := \boldsymbol{N}^{n(\frac{1}{p_2}-\frac{1}{p})}\boldsymbol{\sigma}.$$

By Proposition 3.1,

$$B^{\boldsymbol{\sigma}',\boldsymbol{N}}_{p_1,p} \hookrightarrow F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\sigma}'',\boldsymbol{N}}_{p_2,p}. \tag{36}$$

Let Λ be an admissible function with $\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, and let Λ' and Λ'' be defined by

$$\Lambda'(z) := \Lambda(z) z^{n(\frac{1}{p_1}-\frac{1}{p})} \quad \text{and} \quad \Lambda''(z) := \Lambda(z) z^{n(\frac{1}{p_2}-\frac{1}{p})}, \quad z \in (0,\infty).$$

Those functions are also admissible and, moreover,

$$\Lambda'(z) \sim \sigma'_j \qquad \text{and} \qquad \Lambda''(z) \sim \sigma''_j,\ z \in [N_j, N_{j+1}],\ j \in \mathbb{N}_0,$$

in both cases with equivalence constants independent of j, we refer to [7, Lemma 4.9]. Then, by Proposition 2.11(i), (36) and the previous steps, we obtain, for any $t \in (0, N_1^{-1}]$,

$$\begin{aligned}\mathcal{E}_{\mathsf{C}}^{F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) &\leqslant c_1 \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma}'',\boldsymbol{N}}_{p_2,p}}(t) \leqslant c_2 \left(\int_t^{N_0^{-1}} \Lambda''\big(u^{-1}\big)^{-p'} u^{-(\frac{n}{p_2}+1)p'} \frac{\mathrm{d}u}{u} \right)^{1/p'} \\ &= c_2 \left(\int_t^{N_0^{-1}} \Lambda\big(u^{-1}\big)^{-p'} u^{-(\frac{n}{p}+1)p'} \frac{\mathrm{d}u}{u} \right)^{1/p'} = c_2 \phi^{\boldsymbol{\sigma}}_{p'}(t),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\mathcal{E}_{\mathsf{C}}^{F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) &\geqslant c_1 \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma}',\boldsymbol{N}}_{p_1,p}}(t) \geqslant c_2 \left(\int_t^{N_0^{-1}} \Lambda'\big(u^{-1}\big)^{-p'} u^{-(\frac{n}{p_1}+1)p'} \frac{\mathrm{d}u}{u} \right)^{1/p'} \\ &= c_2 \left(\int_t^{N_0^{-1}} \Lambda\big(u^{-1}\big)^{-p'} u^{-(\frac{n}{p}+1)p'} \frac{\mathrm{d}u}{u} \right)^{1/p'} = c_2 \phi^{\boldsymbol{\sigma}}_{p'}(t)\end{aligned}$$

(with the appropriate modification if $p' = \infty$, i.e., if $0 < p \leqslant 1$), which finishes the proof of (ii). □

Remark 4.2. The necessity of the conditions of Proposition 3.4 are a consequence of the preceding proposition, due to Proposition 2.11(ii) and

$$\int_0^{N_0^{-1}} \Lambda\big(y^{-1}\big)^{-u'} y^{-(\frac{n}{p}+1)u'} \frac{\mathrm{d}y}{y} \sim \sum_{j=0}^{\infty} \big(\sigma_j^{-1} N_j^{\frac{n}{p}+1}\big)^{u'}$$

(with the usual modification if $u' = \infty$).

Theorem 4.3. *Let* $0 < p, q \leqslant \infty$, $\boldsymbol{\sigma}$ *and* $\boldsymbol{N}$ *be admissible sequences, the latter satisfying* $\underline{N}_1 > 1$. *Assume further that* (25) *holds. Let* Λ *be any admissible function such that* $\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, *with equivalence constants independent of* j, *and let* $\phi_u^{\boldsymbol{\sigma}}$ *be defined by* (26) *and* (27).

(i) *Then*

$$\mathfrak{E}_{\mathsf{C}}\big(B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\big) = \big(\phi^{\boldsymbol{\sigma}}_{q'}(t), q\big).$$

(ii) *If* $p < \infty$, *then*

$$\mathfrak{E}_{\mathsf{C}}\big(F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\big) = \big(\phi^{\boldsymbol{\sigma}}_{p'}(t), p\big).$$

Proof. Due to Proposition 4.1 it remains to prove that $u_{B^{\sigma,N}_{p,q}} = q$ and $u_{F^{\sigma,N}_{p,q}} = p$.

Step 1. We want to show that $u_{B^{\sigma,N}_{p,q}} \leqslant q$, in case this index exists. The case $q = \infty$ follows immediately from Proposition 4.1, as for some chosen $\varepsilon \in (0, N_1^{-1}]$

$$\mathcal{E}_{\mathsf{C}}^{B^{\sigma,N}_{p,q}}(t) \sim \phi^{\boldsymbol{\sigma}}_{q'}(t), \quad t \in (0, \varepsilon],$$

which implies, for some $c > 0$,

$$\sup_{0<t\leqslant\varepsilon} \frac{\omega(f,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)} \leqslant c\big\| f \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\big\|, \quad f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}.$$

Consider now $0 < q < \infty$. It is enough to prove that for some $\varepsilon \in (0, N_1^{-1}]$ and $c > 0$,

$$\left(\int_0^{\varepsilon} \left(\frac{\omega(f,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)}\right)^q \mu^{\boldsymbol{\sigma}}_{q'}(\mathrm{d}t)\right)^{1/q} \leqslant c\big\| f \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\big\|, \quad f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}, \tag{37}$$

where $\mu^{\boldsymbol{\sigma}}_{q'}$ stands for the Borel measure associated with $-\log\phi^{\boldsymbol{\sigma}}_{q'}$ in $(0,\varepsilon]$.

Consider first the case $1 < q < \infty$. Note that, there are positive constants d_1, d_2 such that, for any $j \in \mathbb{N}$,

$$d_1\left(\sum_{\ell=1}^{j} \sigma_\ell^{-q'} N_\ell^{(\frac{n}{p}+1)q'}\right)^{1/q'} \leqslant \phi^{\boldsymbol{\sigma}}_{q'}\big(N_j^{-1}\big) \leqslant d_2\left(\sum_{\ell=1}^{j} \sigma_\ell^{-q'} N_\ell^{(\frac{n}{p}+1)q'}\right)^{1/q'}. \tag{38}$$

Using the properties of the measure $\mu^{\boldsymbol{\sigma}}_{q'}$ ($\phi^{\boldsymbol{\sigma}}_{q'}$ is continuously differentiable in this case, see [5, Lemma 2.5] for a similar assertion), Marchaud's inequality (13) (with $k \in \mathbb{N}$ as in the proof of Proposition 3.4), a generalized Hardy's inequality (cf. [14, p. 247]) as well as the properties of Λ, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$, we get for any $f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$,

$$\begin{aligned}
&\left(\int_0^{N_1^{-1}} \left(\frac{\omega(f,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)}\right)^q \mu^{\boldsymbol{\sigma}}_{q'}(\mathrm{d}t)\right)^{1/q} \\
&\leqslant c_1\left(\sum_{j=1}^{\infty}\int_{N_{j+1}^{-1}}^{N_j^{-1}} \left(\frac{\omega(f,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)}\right)^q \Lambda\big(t^{-1}\big)^{-q'} t^{-(\frac{n}{p}+1)q'-1}\phi^{\boldsymbol{\sigma}}_{q'}(t)^{-q'}\,\mathrm{d}t\right)^{1/q} \\
&\leqslant c_2\left(\sum_{j=1}^{\infty}\int_{N_{j+1}^{-1}}^{N_j^{-1}} \left(\frac{\int_t^\infty \frac{\omega_k(f,u)}{u}\frac{\mathrm{d}u}{u}}{\phi^{\boldsymbol{\sigma}}_{q'}(t)^{q'}}\right)^q \Lambda\big(t^{-1}\big)^{-q'} t^{-(\frac{n}{p}+1)q'-1}\,\mathrm{d}t\right)^{1/q} \\
&\leqslant c_3\left(\sum_{j=1}^{\infty}\left(\frac{\int_{N_{j+1}^{-1}}^{N_2^{-1}} \frac{\omega_k(f,u)}{u}\frac{\mathrm{d}u}{u} + \int_{N_2^{-1}}^\infty \frac{\omega_k(f,u)}{u}\frac{\mathrm{d}u}{u}}{\phi^{\boldsymbol{\sigma}}_{q'}(N_j^{-1})^{q'}}\right)^q \sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'}\right)^{1/q} \\
&\leqslant c_4\left(\sum_{j=1}^{\infty}\sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'}\left(\frac{\|f|C\| + \sum_{\ell=2}^{j}\int_{N_{\ell+1}^{-1}}^{N_\ell^{-1}} \frac{\omega_k(f,u)}{u}\frac{\mathrm{d}u}{u}}{\sum_{\ell=1}^{j}\sigma_\ell^{-q'}N_\ell^{(\frac{n}{p}+1)q'}}\right)^q\right)^{1/q} \\
&\leqslant c_5\left(\sum_{j=1}^{\infty}\sigma_j^{-q'} N_j^{(\frac{n}{p}+1)q'}\left(\frac{\|f|C\| + \sum_{\ell=2}^{j}\frac{\omega_k(f,N_\ell^{-1})}{N_\ell^{-1}}}{\sum_{\ell=1}^{j}\sigma_\ell^{-q'}N_\ell^{(\frac{n}{p}+1)q'}}\right)^q\right)^{1/q} \\
&\leqslant c_6\left(\|f|C\|^q + \sum_{j=2}^{\infty}\sigma_j^q N_j^{-(\frac{n}{p}+1)q}\frac{\omega_k(f,N_j^{-1})^q}{N_j^{-q}}\right)^{1/q} \\
&\leqslant c_7\left(\|f|C\| + \left(\int_0^{N_1^{-1}}\big(\omega_k(f,u)\Upsilon\big(u^{-1}\big)\big)^q\frac{\mathrm{d}u}{u}\right)^{1/q}\right) \\
&\leqslant c_8\big\|f\big|B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\big\|,
\end{aligned}$$

with Υ as in the proof of Proposition 3.4. We remark that the last inequality follows as in the proof of Proposition 3.4, i.e., by means of Proposition 3.1 and Theorem 2.8.

We assume now that $0 < q \leqslant 1$ and we will prove (37). Note that, by (27), there are positive constants C_1 and C_2 such that

$$C_1 \sup_{l\leqslant j}\sigma_l^{-1}N_l^{\frac{n}{p}+1} \leqslant \phi^{\boldsymbol{\sigma}}_\infty\big(N_j^{-1}\big) \leqslant C_2 \sup_{l\leqslant j}\sigma_l^{-1}N_l^{\frac{n}{p}+1}, \quad j \in \mathbb{N}. \tag{39}$$

Due to the hypothesis (25), $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{\frac{n}{p}+1} \notin \ell_\infty$, which enables us to construct a strictly increasing sequence $(t_j)_{j\in\mathbb{N}_0}$ of natural numbers in the following way:

(i) $t_0 = 1$;
(ii) t_{j+1}, $j \in \mathbb{N}_0$, is the smallest integer satisfying

$$\frac{\sup_{l\leqslant t_{j+1}} \sigma_l^{-1} N_l^{\frac{n}{p}+1}}{\sup_{l\leqslant t_j} \sigma_l^{-1} N_l^{\frac{n}{p}+1}} \geqslant \frac{2C_2}{C_1}, \tag{40}$$

with C_1, C_2 as in (39). Notice that then

$$\frac{\sup_{l\leqslant t_{j+1}} \sigma_l^{-1} N_l^{\frac{n}{p}+1}}{\sup_{l\leqslant t_j} \sigma_l^{-1} N_l^{\frac{n}{p}+1}} \sim \frac{\sup_{l\leqslant t_{j+1}-1} \sigma_l^{-1} N_l^{\frac{n}{p}+1}}{\sup_{l\leqslant t_j} \sigma_l^{-1} N_l^{\frac{n}{p}+1}} < \frac{2C_2}{C_1}, \quad j \in \mathbb{N}_0. \tag{41}$$

Recalling the definition of the measure $\mu_\infty^{\boldsymbol{\sigma}}$, using Marchaud's inequality (13) (with $k \in \mathbb{N}$ as in the proof of Proposition 3.4), (39)–(41), we obtain, for any $f \in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$,

$$\begin{aligned}
&\left(\int_0^{N_1^{-1}} \left(\frac{\omega(f,t)}{t\phi_\infty^{\boldsymbol{\sigma}}(t)} \right)^q \mu_\infty^{\boldsymbol{\sigma}}(\mathrm{d}t) \right)^{1/q} \\
&\leqslant c_1 \left(\sum_{j=0}^{\infty} \int_{N_{t_{j+1}}^{-1}}^{N_{t_j}^{-1}} \left(\frac{\int_t^\infty \frac{\omega_k(f,u)}{u} \frac{\mathrm{d}u}{u}}{\phi_\infty^{\boldsymbol{\sigma}}(t)} \right)^q \mu_\infty^{\boldsymbol{\sigma}}(\mathrm{d}t) \right)^{1/q} \\
&\leqslant c_2 \left(\sum_{j=0}^{\infty} \phi_\infty^{\boldsymbol{\sigma}}\big(N_{t_j}^{-1}\big)^{-q} \left(\|f|C\| + \int_{N_{t_{j+1}}^{-1}}^{N_{t_0}^{-1}} \frac{\omega_k(f,u)}{u} \frac{\mathrm{d}u}{u} \right)^q \mu_\infty^{\boldsymbol{\sigma}}\big[N_{t_{j+1}}^{-1}, N_{t_j}^{-1}\big] \right)^{1/q} \\
&\leqslant c_3 \left(\sum_{j=0}^{\infty} \Big(\sup_{m\leqslant t_j} \sigma_m^{-1} N_m^{\frac{n}{p}+1} \Big)^{-q} \left(\|f|C\| + \sum_{l=t_0}^{t_{j+1}-1} \int_{N_{l+1}^{-1}}^{N_l^{-1}} \frac{\omega_k(f,u)}{u} \frac{\mathrm{d}u}{u} \right)^q \right)^{1/q} \\
&\leqslant c_4 \left(\sum_{j=0}^{\infty} 2^{-jq} \|f|C\|^q + \sum_{j=0}^{\infty} \Big(\sup_{m\leqslant t_j} \sigma_m^{-1} N_m^{\frac{n}{p}+1} \Big)^{-q} \sum_{h=0}^{j} \sum_{l=t_h}^{t_{h+1}-1} \omega_k\big(f, N_l^{-1}\big)^q N_l^q \right)^{1/q} \\
&\leqslant c_5 \left(\|f|C\|^q + \sum_{j=0}^{\infty} \sum_{h=0}^{j} \left(\frac{\sup_{m\leqslant t_h} \sigma_m^{-1} N_m^{\frac{n}{p}+1}}{\sup_{m\leqslant t_j} \sigma_m^{-1} N_m^{\frac{n}{p}+1}} \right)^q \right.
\end{aligned}$$

$$\times \sum_{l=t_h}^{t_{h+1}-1} \Bigl(\sup_{m \leqslant t_h} \sigma_m^{-1} N_m^{\frac{n}{p}+1} \Bigr)^{-q} \omega_k\bigl(f, N_l^{-1}\bigr)^q N_l^q \Biggr)^{1/q}$$

$$\leqslant c_5 \Biggl(\|f|C\|^q + \sum_{j=0}^{\infty} \sum_{h=0}^{j} 2^{-(j-h)q} \sum_{l=t_h}^{t_{h+1}-1} \Biggl(\frac{\sigma_l^{-1} N_l^{\frac{n}{p}+1}}{\sup_{m \leqslant t_h} \sigma_m^{-1} N_m^{\frac{n}{p}+1}} \Biggr)^q \omega_k\bigl(f, N_l^{-1}\bigr)^q \sigma_l^q N_l^{-\frac{n}{p}q} \Biggr)^{1/q}$$

$$\leqslant c_6 \Biggl(\|f|C\| + \Biggl(\sum_{r=0}^{\infty} 2^{-rq} \sum_{l=1}^{\infty} \omega_k\bigl(f, N_l^{-1}\bigr)^q \sigma_l^q N_l^{-\frac{n}{p}q} \Biggr)^{1/q} \Biggr)$$

$$\leqslant c_7 \Biggl(\|f|C\| + \Biggl(\sum_{l=0}^{\infty} \omega_k\bigl(f, N_{l+1}^{-1}\bigr)^q \sigma_l^q N_l^{-\frac{n}{p}q} \Biggr)^{1/q} \Biggr)$$

$$\leqslant c_8 \Biggl(\|f|C\| + \Biggl(\int_0^{N_0^{-1}} \bigl(\Upsilon\bigl(u^{-1}\bigr) \omega_k(f,u) \bigr)^q \frac{\mathrm{d}u}{u} \Biggr)^{1/q} \Biggr)$$

$$\leqslant c_9 \bigl\| f \bigl| B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \bigr\|,$$

with Υ as before.

Step 2. Assume that for some $v \in (0, \infty)$ there is a positive constant $c(v)$ such that

$$\Biggl(\int_0^{\varepsilon} \Biggl(\frac{\omega(f,t)}{t \phi_{q'}^{\boldsymbol{\sigma}}(t)} \Biggr)^v \mu_{q'}^{\boldsymbol{\sigma}}(\mathrm{d}t) \Biggr)^{1/v} \leqslant c(v) \bigl\| f \bigl| B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \bigr\| \quad \text{for all } f \in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}, \tag{42}$$

where $\mu_{q'}^{\boldsymbol{\sigma}}$ is the Borel measure associated with $-\log \phi_{q'}^{\boldsymbol{\sigma}}$ in $(0, \varepsilon]$ for some $\varepsilon \in (0, N_1^{-1}]$.

Let first $1 < q \leqslant \infty$. Due to the hypothesis (25), we can construct a strictly increasing sequence $(t_j)_{j \in \mathbb{N}_0}$ of natural numbers in the following way:

(i) t_0 is such that $N_{t_0}^{-1} \leqslant \varepsilon$;
(ii) t_{j+1}, $j \in \mathbb{N}_0$, is the smallest integer satisfying

$$\frac{\sum_{l=1}^{t_{j+1}} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}}{\sum_{l=1}^{t_j} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}} \geqslant \left(\frac{2d_2}{d_1} \right)^{q'}, \tag{43}$$

with d_1, d_2 as in (38). Notice that then

$$\frac{\sum_{l=1}^{t_{j+1}} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}}{\sum_{l=1}^{t_j} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}} \sim \frac{\sum_{l=1}^{t_{j+1}-1} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}}{\sum_{l=1}^{t_j} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}} < \left(\frac{2d_2}{d_1} \right)^{q'}, \quad j \in \mathbb{N}_0. \tag{44}$$

Given such a sequence $(t_j)_{j \in \mathbb{N}_0}$, for each $J \in \mathbb{N}$ denote by f_J the function $f^{\boldsymbol{b}}$ in (31) with $\boldsymbol{b} = (b_j)_{j \in \mathbb{N}}$ being the sequence defined by

$$b_j := \begin{cases} \sigma_j^{1-q'} N_j^{-(\frac{n}{p}+1)(1-q')} \big(\sum_{l=t_0+1}^{t_k} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'}\big)^{-1/q} & \text{for } j = t_{k-1}+1, \dots, t_k, \\ & \quad k = 1, \dots, J, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $b_j \geqslant 0$, $j \in \mathbb{N}$, $\boldsymbol{b} \in \ell_q$ and, moreover, $\|\boldsymbol{b}|\ell_q\| \leqslant J^{1/q}$, so that (32), (42), (33), (38), (43) and (44), allow us to write, with constants independent of J,

$$\begin{aligned}
J^{1/q} &\geqslant c_1 \big\| f_J \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \big\| \\
&\geqslant c_2 \left(\int_0^{\varepsilon} \left(\frac{\omega(f_J,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)} \right)^v \mu^{\boldsymbol{\sigma}}_{q'}(\mathrm{d}t) \right)^{1/v} \\
&\geqslant c_2 \left(\sum_{j=0}^{J} \int_{N_{t_{j+1}}^{-1}}^{N_{t_j}^{-1}} \left(\frac{\omega(f_J,t)}{t\phi^{\boldsymbol{\sigma}}_{q'}(t)} \right)^v \mu^{\boldsymbol{\sigma}}_{q'}(\mathrm{d}t) \right)^{1/v} \\
&\geqslant c_3 \left(\sum_{j=0}^{J} \left(\frac{\omega(f_J, N_{t_j}^{-1})}{N_{t_j}^{-1} \phi^{\boldsymbol{\sigma}}_{q'}(N_{t_{j+1}}^{-1})} \right)^v \mu^{\boldsymbol{\sigma}}_{q'}\big[N_{t_{j+1}}^{-1}, N_{t_j}^{-1}\big] \right)^{1/v} \\
&\geqslant c_4 \left(\sum_{j=0}^{J} \left(\sum_{m=1}^{t_j} b_m \sigma_m^{-1} N_m^{(\frac{n}{p}+1)} \right)^v \left(\sum_{l=1}^{t_{j+1}} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'} \right)^{-v/q'} \log \frac{\phi^{\boldsymbol{\sigma}}_{q'}(N_{t_{j+1}}^{-1})}{\phi^{\boldsymbol{\sigma}}_{q'}(N_{t_j}^{-1})} \right)^{1/v} \\
&\geqslant c_5 \left(\sum_{j=0}^{J} \left(\sum_{m=t_0+1}^{t_j} \sigma_m^{-q'} N_m^{(\frac{n}{p}+1)q'} \right)^{v/q'} \left(\sum_{l=1}^{t_{j+1}} \sigma_l^{-q'} N_l^{(\frac{n}{p}+1)q'} \right)^{-v/q'} \right)^{1/v} \\
&\geqslant c_6 J^{1/v}.
\end{aligned}$$

This being true for all $J \in \mathbb{N}$ implies $v \geqslant q$.

Now let $0 < q \leqslant 1$. We modify the preceding proof in an appropriate way. The sequence $(t_j)_{j\in\mathbb{N}_0}$ should satisfy (i), but instead of (43) we shall assume that it satisfies the inequality (40). The sequence $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ shall be defined by

$$b_j := \begin{cases} 1, & \text{for } j = t_k,\ k \in \{1, \dots, J\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $b_j \geqslant 0$, $j \in \mathbb{N}$, $\boldsymbol{b} \in \ell_q$ and, moreover, $\|\boldsymbol{b}|\ell_q\| = J^{1/q}$, so that (32), (42), (33) and the properties of the sequence $(t_j)_{j\in\mathbb{N}_0}$, allow us to write,

$$\begin{aligned}
J^{1/q} &\geqslant c_1 \big\| f_J \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \big\| \\
&\geqslant c_2 \left(\int_0^{\varepsilon} \left(\frac{\omega(f_J,t)}{t\phi^{\boldsymbol{\sigma}}_{\infty}(t)} \right)^v \mu^{\boldsymbol{\sigma}}_{\infty}(\mathrm{d}t) \right)^{1/v}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant c_2\left(\sum_{j=0}^{J}\int_{N_{t_{j+1}}^{-1}}^{N_{t_j}^{-1}}\left(\frac{\omega(f_J,t)}{t\phi_\infty^{\boldsymbol{\sigma}}(t)}\right)^v\mu_\infty^{\boldsymbol{\sigma}}(\mathrm{d}t)\right)^{1/v}\\
&\geqslant c_3\left(\sum_{j=0}^{J}\left(\frac{\omega(f_J,N_{t_j}^{-1})}{N_{t_j}^{-1}\phi_\infty^{\boldsymbol{\sigma}}(N_{t_{j+1}}^{-1})}\right)^v\mu_\infty^{\boldsymbol{\sigma}}\left[N_{t_{j+1}}^{-1},N_{t_j}^{-1}\right]\right)^{1/v}\\
&\geqslant c_4\left(\sum_{j=0}^{J}\left(\frac{\sum_{m=0}^{j}\sigma_{t_m}^{-1}N_{t_m}^{(\frac{n}{p}+1)}}{\sup_{l\leqslant t_{j+1}}\sigma_l^{-1}N_l^{\frac{n}{p}+1}}\right)^v\log\frac{\phi_\infty^{\boldsymbol{\sigma}}(N_{t_{j+1}}^{-1})}{\phi_\infty^{\boldsymbol{\sigma}}(N_{t_j}^{-1})}\right)^{1/v}\\
&\geqslant c_5\left(\sum_{j=0}^{J}\left(\frac{\sup_{m=1,\dots,j}\sigma_{t_m}^{-1}N_{t_m}^{(\frac{n}{p}+1)}}{\sup_{l\leqslant t_{j+1}}\sigma_l^{-1}N_l^{\frac{n}{p}+1}}\right)^v\right)^{1/v}\\
&\geqslant c_6J^{1/v},
\end{aligned}
$$

with constants independent of J. This being true for all $J\in\mathbb{N}$, implies $v\geqslant q$. Then we have shown that $u_{B_{p,q}^{\sigma,N}}\geqslant q$, completing the proof of (i).

Step 3. Part (ii) is a direct consequence of part (i), taking into account the embeddings in (36) and Proposition 2.11(iii). □

Remark 4.4. Theorem 4.3 generalizes the results previously obtained by Caetano and Haroske [6, Theorem 3.1], which already extend the ones obtained by Haroske and Moura [17, Theorem 3.1], both for the spaces of generalized smoothness $A_{pq}^{(s,\Psi)}$ (see Remark 2.7). The latter cover also the results obtained by Haroske and Triebel for the classical Besov and Triebel–Lizorkin spaces A_{pq}^s, cf. [16,27].

5. Applications

As a first application of Theorem 4.3 and (16) we obtain some Hardy type inequalities.

Corollary 5.1. *Consider the same hypotheses as in Theorem* 4.3 *and let* $\varepsilon\in(0,N_1^{-1}]$.

(i) *Let* κ *be a positive monotonically decreasing function on* $(0,\varepsilon]$ *and let* $0<u\leqslant\infty$. *Then*

$$
\left(\int_0^\varepsilon\left(\kappa(t)\frac{\omega(f,t)}{t\phi_{q'}^{\boldsymbol{\sigma}}(t)}\right)^u\mu_{q'}^{\boldsymbol{\sigma}}(\mathrm{d}t)\right)^{1/u}\leqslant c\left\|f\,\middle|\,B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}\right\| \tag{45}
$$

for some $c>0$ *and all* $f\in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$ *if, and only if,* κ *is bounded and* $q\leqslant u\leqslant\infty$, *with the modification*

$$
\sup_{t\in(0,\varepsilon)}\kappa(t)\frac{\omega(f,t)}{t\phi_{q'}^{\boldsymbol{\sigma}}(t)}\leqslant c\left\|f\,\middle|\,B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}\right\| \tag{46}
$$

if $u=\infty$. In particular, if κ is an arbitrary non-negative function on $(0,\varepsilon]$*, then* (46) *holds if, and only if, κ is bounded.*

(ii) *Let κ be a positive monotonically decreasing function on* $(0,\varepsilon]$ *and let* $0<u\leqslant\infty$*. Then*

$$\left(\int_0^{\varepsilon}\left(\kappa(t)\frac{\omega(f,t)}{t\phi_{p'}^{\boldsymbol{\sigma}}(t)}\right)^u \mu_{p'}^{\boldsymbol{\sigma}}(\mathrm{d}t)\right)^{1/u} \leqslant c\left\| f \,\middle|\, F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}\right\| \tag{47}$$

for some $c>0$ and all $f\in F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$ if, and only if, κ is bounded and $p\leqslant u\leqslant\infty$, with the modification

$$\sup_{t\in(0,\varepsilon)}\kappa(t)\frac{\omega(f,t)}{t\phi_{p'}^{\boldsymbol{\sigma}}(t)} \leqslant c\left\| f \,\middle|\, F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}\right\| \tag{48}$$

if $u=\infty$. In particular, if κ is an arbitrary non-negative function on $(0,\varepsilon]$*, then* (48) *holds if, and only if, κ is bounded.*

Remark 5.2. Observe that, when $1<q\leqslant\infty$, the measure $\mu_{q'}^{\boldsymbol{\sigma}}(\mathrm{d}t)$ in (45) can be replaced by

$$\frac{\mathrm{d}t}{\phi_{q'}^{\boldsymbol{\sigma}}(t)^{q'}t^{(\frac{n}{p}+1)q'+1}\Lambda(t^{-1})^{q'}}.$$

An analogous replacement can be done for the measure $\mu_{p'}^{\boldsymbol{\sigma}}(\mathrm{d}t)$ in (47), if $1<p<\infty$.

Another application concerns embedding assertions of the type

$$A_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow A_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}} \tag{49}$$

with $0<p_1\leqslant p_2\leqslant\infty$, $0<q_1,q_2\leqslant\infty$, and $\boldsymbol{\sigma},\boldsymbol{\tau}$ admissible.

We shall be mainly concerned with 'limiting' situations and want to explain first what we shall understand by this. Recall that in case of embeddings of spaces $A_{p,q}^s$ or $A_{p,q}^{(s,\Psi)}$, respectively, the counterpart of (49) is always true for $s_1-\frac{n}{p_1}>s_2-\frac{n}{p_2}$ (where $s_1\geqslant s_2$, $0<p_1\leqslant p_2\leqslant\infty$), irrespective of the parameters q_i (and Ψ_i), whereas they influence the 'limiting' situation $s_1-\frac{n}{p_1}=s_2-\frac{n}{p_2}$. In view of Remark 2.7 our notion of 'limiting' situation should naturally include this case (i.e., when $\boldsymbol{\sigma}=(2^{js})_{j\in\mathbb{N}_0}$, $\boldsymbol{N}=(2^j)_{j\in\mathbb{N}_0}$). Following that idea, it seems reasonable to consider the corresponding indices of the sequences $\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}$ and $\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}$, but one immediately faces the difficulty to decide which to choose: Assume that $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1})>\bar{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$, then

$$\underline{s}\left(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1}\boldsymbol{N}^{-\frac{n}{p_1}+\frac{n}{p_2}}\right)\geqslant\underline{s}\left(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\right)-\bar{s}\left(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\right)>0,$$

such that $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{\frac{n}{p_1}-\frac{n}{p_2}}\in\ell_v$ for any $v\in(0,\infty]$, see Remark 2.1. Then due to the result of Caetano and Farkas in [5, Theorem 3.7], $B_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}}\hookrightarrow B_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}}$ for any choice of $q_1,q_2\in(0,\infty]$, see Remark 3.2. Consequently, $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1})=\bar{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$ appears as obvious limiting case to study.

On the other hand, due to the standardization result [7, Theorem 1] one can identify spaces of type $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ with some suitable space $A^{\boldsymbol{\beta}}_{p,q}$ (neglecting technical details), and for embeddings of such spaces one immediately has a necessary criterion, too. Note that

$$A^{\bar{s}(\boldsymbol{\beta})+\varepsilon}_{p,q} \hookrightarrow A^{\boldsymbol{\beta}}_{p,q} \hookrightarrow A^{\underline{s}(\boldsymbol{\beta})-\varepsilon}_{p,q},$$

for arbitrary $\varepsilon > 0$, see (10). Thus $A^{\boldsymbol{\sigma}'}_{p_1,q_1} \hookrightarrow A^{\boldsymbol{\tau}'}_{p_2,q_2}$ implies $\bar{s}(\boldsymbol{\sigma}') \geqslant \underline{s}(\boldsymbol{\tau}')$. Returning to our problem, also $\bar{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}) = \underline{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$ is entitled to be called a 'limiting' case.

Here we consider the limiting case $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}) = \bar{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$ and refine the result of Caetano and Farkas in [5, Theorem 3.7], see Remark 3.2, as follows.

Theorem 5.3. *Let* $0 < q_1, q_2 \leqslant \infty$, *and* $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ *be admissible sequences,* $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ *admissible with* $\underline{N}_1 > 1$.

(i) *Let* $0 < p_1 \leqslant p_2 \leqslant \infty$, *and assume that*

$$\underline{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\big) = \bar{s}\big(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\big), \tag{50}$$

and

$$0 \leqslant \bar{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\big) - \underline{s}\big(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\big) < \underline{s}(\boldsymbol{N}). \tag{51}$$

Then

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2} \tag{52}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p_2})} \in \ell_{q^*}, \tag{53}$$

where q^* *is given by*

$$\frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+. \tag{54}$$

(ii) *Let* $0 < p_1 < p_2 < \infty$, *and assume* (50) *and* (51). *Then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \hookrightarrow F^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2} \tag{55}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p_2})} \in \ell_\infty. \tag{56}$$

(iii) *Let* $0 < p < \infty$, *and assume that* (50) *and* (51) *are satisfied with* $p_1 = p_2 = p$. *Then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q_1} \hookrightarrow F^{\boldsymbol{\tau},\boldsymbol{N}}_{p,q_2} \tag{57}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau} \in \ell_{q^*}, \tag{58}$$

where q^* is given by (54).

Proof.

Step 1. The sufficiency in (i) is covered by [5, Theorem 3.7], it remains to prove the necessity of (53). First we use some lift argument: let $\boldsymbol{\nu}$ be an admissible sequence, $\varphi^{\boldsymbol{N}}$ a system of functions as in Definition 2.4, and the operator $I_{\boldsymbol{\nu}}$ be given by

$$I_{\boldsymbol{\nu}} f = \left(\sum_{j=0}^{\infty} \nu_j \varphi_j^{\boldsymbol{N}} \widehat{f} \right)^{\vee}, \quad f \in \mathcal{S}', \tag{59}$$

then $I_{\boldsymbol{\nu}}$ maps $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$ isomorphically onto $A_{p,q}^{\boldsymbol{\sigma}\boldsymbol{\nu}^{-1},\boldsymbol{N}}$; cf. [13, Theorem 3.1.9, Definition 3.2.3]. We claim that we can choose $\boldsymbol{\nu}$ appropriately to obtain

$$\widetilde{\boldsymbol{\sigma}}^{-1}\boldsymbol{N}^{\frac{n}{p_1}+1} \notin \ell_{q_1'}, \quad \text{and} \quad \underline{s}\big(\widetilde{\boldsymbol{\sigma}}\boldsymbol{N}^{-n/p_1}\big) \geqslant \underline{s}\big(\widetilde{\boldsymbol{\tau}}\boldsymbol{N}^{-n/p_2}\big) > 0, \tag{60}$$

where $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\boldsymbol{\nu}^{-1}$, $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}\boldsymbol{\nu}^{-1}$. This can be seen as follows. By our assumption (51), $\bar{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}) - \underline{s}(\boldsymbol{N}) < \underline{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$ and we can choose some admissible sequence $\boldsymbol{\nu}$ such that

$$\bar{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}}\big) - \underline{s}(\boldsymbol{N}) < \underline{s}(\boldsymbol{\nu}) \leqslant \bar{s}(\boldsymbol{\nu}) < \underline{s}\big(\boldsymbol{\tau}\boldsymbol{N}^{-\frac{n}{p_2}}\big). \tag{61}$$

The left-hand inequality of (61) leads to

$$\bar{s}\big(\boldsymbol{\nu}^{-1}\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}-1}\big) \leqslant -\underline{s}(\boldsymbol{\nu}) - \underline{s}(\boldsymbol{N}) + \bar{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}}\big) < 0,$$

and Remark 2.1 implies $\widetilde{\boldsymbol{\sigma}}^{-1}\boldsymbol{N}^{n/p_1+1} = \boldsymbol{\nu}\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p_1+1} \notin \ell_\infty$. Likewise the right-hand side of (61) implies $\underline{s}(\widetilde{\boldsymbol{\tau}}\boldsymbol{N}^{-n/p_2}) = \underline{s}(\boldsymbol{\nu}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}) > 0$ and (50) guarantees $\underline{s}(\widetilde{\boldsymbol{\sigma}}\boldsymbol{N}^{-n/p_1}) \geqslant \underline{s}(\widetilde{\boldsymbol{\tau}}\boldsymbol{N}^{-n/p_2})$. Moreover, by the properties of $I_{\boldsymbol{\nu}}$, embedding (52) exists if, and only if,

$$B_{p_1,q_1}^{\widetilde{\boldsymbol{\sigma}},\boldsymbol{N}} \hookrightarrow B_{p_2,q_2}^{\widetilde{\boldsymbol{\tau}},\boldsymbol{N}}. \tag{62}$$

Without loss of generality we shall thus assume that (60) holds with $\widetilde{\boldsymbol{\sigma}}$ and $\widetilde{\boldsymbol{\tau}}$ replaced by $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, respectively. Consequently, an application of Proposition 3.4 gives $\boldsymbol{\tau}^{-1}\boldsymbol{N}^{n/p_2+1} \notin \ell_{q_2'}$ in view of (52) and (60). Hence both involved spaces $B_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}}$, $B_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}}$ possess non-trivial continuity envelope functions, and (52) implies the existence of some $c > 0$ such that for all $0 < t < 1$,

$$\mathcal{E}_{\mathsf{C}}^{B_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}}}(t) \leqslant c\,\mathcal{E}_{\mathsf{C}}^{B_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}}}(t) \tag{63}$$

by Proposition 2.11(i). We apply Proposition 4.1(i),

$$\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1}}\left(N_j^{-1}\right) \sim \phi^{\boldsymbol{\sigma}}_{q_1'}\left(N_j^{-1}\right) \sim \left(\sum_{k=1}^{j} \sigma_k^{-q_1'} N_k^{(\frac{n}{p_1}+1)q_1'}\right)^{1/q_1'}, \quad j \in \mathbb{N}, \tag{64}$$

and

$$\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2}}\left(N_j^{-1}\right) \sim \phi^{\boldsymbol{\tau}}_{q_2'}\left(N_j^{-1}\right) \sim \left(\sum_{k=1}^{j} \tau_k^{-q_2'} N_k^{(\frac{n}{p_2}+1)q_2'}\right)^{1/q_2'}, \quad j \in \mathbb{N}, \tag{65}$$

see also (38).

Step 2. We prove that (52) necessarily implies $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(1/p_1-1/p_2)} \in \ell_\infty$. For that reason we construct a special sequence $\boldsymbol{\nu} = (\nu_k)_{k\in\mathbb{N}_0}$ and use (62), (64) and (65). Choose ε such that

$$\bar{s}\left(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\right) - \underline{s}\left(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\right) < \varepsilon < \underline{s}(\boldsymbol{N}).$$

This is possible due to (51). We claim that $\boldsymbol{\nu}$, given by

$$\nu_k = 2^{k\varepsilon}\tau_k N_k^{-(\frac{n}{p_2}+1)}, \quad k \in \mathbb{N}_0, \tag{66}$$

satisfies (60). In order to show the right-hand side of (60) it is sufficient to note that

$$\underline{s}\left(\boldsymbol{\tau}\boldsymbol{\nu}^{-1}\boldsymbol{N}^{-\frac{n}{p_2}}\right) \geqslant \underline{s}(\boldsymbol{N}) - \varepsilon > 0.$$

Concerning the left-hand assertion in (60) it follows from

$$\begin{aligned}\bar{s}\left(\boldsymbol{\sigma}\boldsymbol{\nu}^{-1}\boldsymbol{N}^{-\frac{n}{p_1}-1}\right) &\leqslant \bar{s}\left(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1}\boldsymbol{N}^{n(\frac{1}{p_2}-\frac{1}{p_1})}\right) - \varepsilon \\ &\leqslant \bar{s}\left(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\right) - \underline{s}\left(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\right) - \varepsilon < 0.\end{aligned}$$

We use (64), (65) for the lifted embedding (62) with the special setting $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\boldsymbol{\nu}^{-1}$, $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}\boldsymbol{\nu}^{-1}$, and $\boldsymbol{\nu}$ given by (66) now. Consequently,

$$\begin{aligned}\left(\sum_{k=1}^{j} \nu_k^{q_1'}\sigma_k^{-q_1'} N_k^{(\frac{n}{p_1}+1)q_1'}\right)^{1/q_1'} &\leqslant c\left(\sum_{k=1}^{j} \nu_k^{q_2'}\tau_k^{-q_2'} N_k^{(\frac{n}{p_2}+1)q_2'}\right)^{1/q_2'} \\ &= c\left(\sum_{k=1}^{j} 2^{k\varepsilon q_2'}\right)^{1/q_2'} \leqslant c' 2^{j\varepsilon}\end{aligned}$$

for some $c' > 0$ and all $j \in \mathbb{N}$ (appropriately modified for $q_2' = \infty$). The left-hand side can be further estimated from below by

$$\nu_j\sigma_j^{-1}N_j^{(\frac{n}{p_1}+1)} = 2^{j\varepsilon}\,\sigma_j^{-1}\tau_j N_j^{n(\frac{1}{p_1}-\frac{1}{p_2})}, \quad j \in \mathbb{N},$$

leading to

$$\sigma_j^{-1}\tau_j N_j^{n(\frac{1}{p_1}-\frac{1}{p_2})} \leqslant c, \quad j \in \mathbb{N},$$

that is, $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(1/p_1-1/p_2)} \in \ell_\infty$, completing the proof of (53) in case of $q_1 \leqslant q_2$.

Step 3. Let now $q_1 > q_2$, then $\frac{1}{q^*} = \frac{1}{q_2} - \frac{1}{q_1}$. Assume first $q_2 > 1$, and put $r := \frac{q_2'}{q_1'} > 1$, then (63), (64) and (65) can be rewritten as

$$\left(\sum_{k=1}^{j} \sigma_k^{-q_1'} N_k^{(\frac{n}{p_1}+1)q_1'}\right)^{1/q_1'} \leqslant c\left(\sum_{k=1}^{j} \left(\tau_k^{-q_1'} N_k^{(\frac{n}{p_2}+1)q_1'}\right)^r\right)^{1/rq_1'} \tag{67}$$

for some $c > 0$ and all $j \in \mathbb{N}$. Let

$$\alpha_k := \sigma_k^{-q_1'} \tau_k^{q_1'} N_k^{(\frac{n}{p_1}-\frac{n}{p_2})q_1'}, \qquad \beta_k := \tau_k^{-q_1'} N_k^{(\frac{n}{p_2}+1)q_1'}, \quad k \in \mathbb{N},$$

then (67) gives

$$\sum_{k=1}^{j} \alpha_k \beta_k \leqslant c' \left(\sum_{k=1}^{j} \beta_k^r\right)^{1/r} \tag{68}$$

for all $j \in \mathbb{N}$. By the above lifting argument this is obviously true for all $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ (in adapted notation). We apply an inequality of Landau, see [14, Theorem 161, p. 120], stating that whenever $\sum_k \alpha_k \beta_k$ is convergent for all sequences $(\beta_k)_{k\in\mathbb{N}}$, for which $\sum_k \beta_k^r$ is convergent, then also $\sum_k \alpha_k^{r'}$ is convergent, assuming $r \in (1,\infty)$ and $\alpha_k, \beta_k \geqslant 0$. Consequently, $(\alpha_k)_{k\in\mathbb{N}} \in \ell_{r'}$, that is

$$\begin{aligned}\left(\sum_{k=1}^{j} \alpha_k^{r'}\right)^{1/r'} &= \left(\sum_{k=1}^{j} \sigma_k^{-q_1' r'} \tau_k^{q_1' r'} N_k^{(\frac{n}{p_1}-\frac{n}{p_2})q_1' r'}\right)^{1/r'} \\ &= \left(\sum_{k=1}^{j} \sigma_k^{-q^*} \tau_k^{q^*} N_k^{(\frac{n}{p_1}-\frac{n}{p_2})q^*}\right)^{q_1'/q^*} \leqslant c\end{aligned}$$

for some $c > 0$ and all $j \in \mathbb{N}$. We thus arrive at (53) with (54).

Let $0 < q_2 \leqslant 1$. In view of (51) we can choose some ε with

$$\bar{s}\left(\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}}\right) - \underline{s}\left(\boldsymbol{\tau}\boldsymbol{N}^{-\frac{n}{p_2}}\right) < (1-\varepsilon)\underline{s}(\boldsymbol{N}) < \underline{s}(\boldsymbol{N}).$$

Then $\boldsymbol{\nu} = \boldsymbol{\tau}\boldsymbol{N}^{-\frac{n}{p_2}-\varepsilon}$ satisfies (60), since $\underline{s}(\boldsymbol{\tau}\boldsymbol{\nu}^{-1}\boldsymbol{N}^{-n/p_2}) = \varepsilon\underline{s}(\boldsymbol{N}) > 0$, and

$$\bar{s}\left(\boldsymbol{\sigma}\boldsymbol{\nu}^{-1}\boldsymbol{N}^{-\frac{n}{p_1}-1}\right) \leqslant \bar{s}\left(\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}}\right) - \underline{s}\left(\boldsymbol{\tau}\boldsymbol{N}^{-\frac{n}{p_2}}\right) - (1-\varepsilon)\underline{s}(\boldsymbol{N}) < 0.$$

Let $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\boldsymbol{\nu}^{-1}$, $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}\boldsymbol{\nu}^{-1}$; then $\underline{N}_1 > 1$ implies

$$\sup_{j\leqslant k} \widetilde{\tau}_j^{-1} N_j^{\frac{n}{p_2}+1} = \sup_{j\leqslant k} N_j^{1-\varepsilon} = N_k^{1-\varepsilon}, \quad k \in \mathbb{N},$$

and, by (39), Proposition 4.1(i) and $0 < q_2 \leqslant 1$,

$$\mathcal{E}_{\mathsf{C}}^{B^{\widetilde{\boldsymbol{\tau}},\boldsymbol{N}}_{p_2,q_2}}\big(N_k^{-1}\big) \sim \phi_\infty^{\widetilde{\boldsymbol{\tau}}}\big(N_k^{-1}\big) \sim N_k^{1-\varepsilon}, \quad k \in \mathbb{N}. \tag{69}$$

We use the example $f^{\boldsymbol{b}}$ given by (an adapted version of) (31) with

$$f^{\boldsymbol{b}}(x) := \sum_{j=1}^{\infty} b_j \widetilde{\sigma}_j^{-1} N_j^{n/p_1} \phi^{\delta,L}(N_j x), \quad x \in \mathbb{R}^n,$$

where $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ is a sequence of non-negative numbers in ℓ_{q_1}, and $\phi^{\delta,L}$ as in (30). According to (32) and (33) we have

$$\big\| f^{\boldsymbol{b}} \big| B^{\widetilde{\boldsymbol{\sigma}},\boldsymbol{N}}_{p_1,q_1} \big\| \leqslant c \|\boldsymbol{b}|\ell_{q_1}\|, \tag{70}$$

and

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \geqslant c \sum_{j=1}^{k} b_j \widetilde{\sigma}_j^{-1} N_j^{n/p_1+1}, \quad k \in \mathbb{N}. \tag{71}$$

Note that (52) implies (62). We apply Theorem 4.3(i) to $B^{\widetilde{\boldsymbol{\tau}},\boldsymbol{N}}_{p_2,q_2}$ and can thus estimate (in a similar way to Step 2 in the proof of Theorem 4.3),

$$\begin{aligned}
\|\boldsymbol{b}|\ell_{q_1}\| &\geqslant c_1 \big\| f^{\boldsymbol{b}} \big| B^{\widetilde{\boldsymbol{\sigma}},\boldsymbol{N}}_{p_1,q_1} \big\| \\
&\geqslant c_2 \big\| f^{\boldsymbol{b}} \big| B^{\widetilde{\boldsymbol{\tau}},\boldsymbol{N}}_{p_2,q_2} \big\| \\
&\geqslant c_3 \left(\int_0^{\varepsilon} \left[\frac{\omega(f^{\boldsymbol{b}}, t)}{t \phi_\infty^{\widetilde{\boldsymbol{\tau}}}(t)} \right]^{q_2} \mu_\infty^{\widetilde{\boldsymbol{\tau}}}(\mathrm{d}t) \right)^{1/q_2} \\
&\geqslant c_4 \left(\sum_{k=J}^{\infty} \left[\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \right]^{q_2} \phi_\infty^{\widetilde{\boldsymbol{\tau}}}\big(N_k^{-1}\big)^{-q_2} \right)^{1/q_2} \\
&\geqslant c_5 \left(\sum_{k=J}^{\infty} \left(\sum_{j=1}^{k} b_j \widetilde{\sigma}_j^{-1} N_j^{n/p_1+1} \right)^{q_2} N_k^{-(1-\varepsilon)q_2} \right)^{1/q_2},
\end{aligned}$$

where we used (69)–(71), and $J \in \mathbb{N}$ is chosen such that $N_J^{-1} < \varepsilon$. Consequently, the special choice of $\boldsymbol{\nu}$ leads to

$$\|\boldsymbol{b}|\ell_{q_1}\| \geqslant c \left(\sum_{k=J}^{\infty} \big[b_k \sigma_k^{-1} \tau_k N_k^{\frac{n}{p_1} - \frac{n}{p_2}} \big]^{q_2} \right)^{1/q_2}. \tag{72}$$

When $q_1 = \infty$, put $b_j \equiv 0$, $j = 1, \ldots, J-1$, $b_j := 1$, $j \geqslant J$. Then (72) gives

$$\left\| \boldsymbol{\sigma}^{-1} \boldsymbol{\tau} \boldsymbol{N}^{\frac{n}{p_1} - \frac{n}{p_2}} \middle| \ell_{q_2} \right\| \leqslant c',$$

coinciding with $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n/p_1 - n/p_2} \in \ell_{q^*}$ in that case. Assume $q_1 < \infty$ now and recall that $r := \frac{q_1}{q_2} > 1$. Then we can reformulate (72) as

$$\sum_{k=J}^{\infty} b_k^{q_2} \left[\sigma_k^{-1} \tau_k N_k^{\frac{n}{p_1} - \frac{n}{p_2}}\right]^{q_2} \leqslant c \left(\sum_{k=J}^{\infty} \left(b_k^{q_2}\right)^r \right)^{1/r}, \tag{73}$$

that is, (68) with

$$\beta_k := b_k^{q_2}, \qquad \alpha_k := \left[\sigma_k^{-1} \tau_k N_k^{\frac{n}{p_1} - \frac{n}{p_2}}\right]^{q_2}, \quad k \in \mathbb{N},$$

where we may assume, for convenience, that $\alpha_k = \beta_k = 0$ when $k < J$. We apply the above-mentioned inequality of Landau again, and obtain that

$$\sum_{k=J}^{\infty} \alpha_k^{r'} = \sum_{k=J}^{\infty} \left[\sigma_k^{-1} \tau_k N_k^{\frac{n}{p_1} - \frac{n}{p_2}}\right]^{r' q_2} = \sum_{k=J}^{\infty} \left[\sigma_k^{-1} \tau_k N_k^{\frac{n}{p_1} - \frac{n}{p_2}}\right]^{q^*}$$

converges, i.e., (53). This completes the proof of (i).

Step 4. We turn to the F-case in (ii) and begin with the sufficiency. Let (56) be satisfied. Due to $p_1 < p_2$ we can choose r_1 and r_2 with $p_1 < r_1 \leqslant r_2 < p_2$ and put

$$\boldsymbol{\gamma} := \boldsymbol{\sigma} \boldsymbol{N}^{n(\frac{1}{r_1} - \frac{1}{p_1})}, \qquad \boldsymbol{\varrho} := \boldsymbol{\tau} \boldsymbol{N}^{n(\frac{1}{r_2} - \frac{1}{p_2})} \tag{74}$$

such that (36) gives

$$F_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow B_{r_1,p_1}^{\boldsymbol{\gamma},\boldsymbol{N}}, \qquad B_{r_2,p_2}^{\boldsymbol{\varrho},\boldsymbol{N}} \hookrightarrow F_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}}. \tag{75}$$

Clearly, (50) and (74) then lead to

$$\underline{s}\left(\boldsymbol{\gamma} N^{-n/r_1}\right) = \underline{s}\left(\boldsymbol{\sigma} N^{-n/p_1}\right) = \bar{s}\left(\boldsymbol{\tau} N^{-n/p_2}\right) = \bar{s}\left(\boldsymbol{\varrho} N^{-n/r_2}\right), \tag{76}$$

similarly for (51), such that (i) and (75) yield $F_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow F_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}}$ under the assumption

$$\boldsymbol{\sigma}^{-1} \boldsymbol{\tau} \boldsymbol{N}^{n(\frac{1}{p_1} - \frac{1}{p_2})} = \boldsymbol{\gamma}^{-1} \boldsymbol{\varrho} \boldsymbol{N}^{n(\frac{1}{r_1} - \frac{1}{r_2})} \in \ell_{p^*} \quad \text{with } \frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1} \right)_+; \tag{77}$$

in view of $p_1 < p_2$ this coincides with (56). Conversely, let (55) be true, then in a similar way as above we choose $r_1 < p_1 \leqslant p_2 < r_2$, $\boldsymbol{\gamma}$ and $\boldsymbol{\varrho}$ as in (74), such that (36) gives

$$B_{r_1,p_1}^{\boldsymbol{\gamma},\boldsymbol{N}} \hookrightarrow F_{p_1,q_1}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow F_{p_2,q_2}^{\boldsymbol{\tau},\boldsymbol{N}} \hookrightarrow B_{r_2,p_2}^{\boldsymbol{\varrho},\boldsymbol{N}}. \tag{78}$$

Together with (76) and (77), part (i) thus completes the proof of (ii).

Step 5. Note that the necessity in case of $q_1 \leqslant q_2$ is already covered by the end of Step 4. Hence it remains to verify the sufficiency of (58) for (57) as well as the necessity in case of $q_1 > q_2$. For the sufficiency we proceed in a similar way as in [23, Theorem 1.1.13(vi)]. Let $f \in F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q_1}$, then by Definition 2.5(ii),

$$\begin{aligned}\|f|F^{\boldsymbol{\tau},\boldsymbol{N}}_{p,q_2}\| &= \left\| \left(\sum_{j=0}^{\infty} \tau_j^{q_2} \left| \left(\varphi_j^N \widehat{f}\right)^{\vee}(\cdot)\right|^{q_2} \right)^{1/q_2} \Big| L_p \right\| \\ &\leqslant \|\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}|\ell_{q^*}\| \left\| \left(\sum_{j=0}^{\infty} \sigma_j^{q_1} \left| \left(\varphi_j^N \widehat{f}\right)^{\vee}(\cdot)\right|^{q_1} \right)^{1/q_1} \Big| L_p \right\| \\ &= \|\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}|\ell_{q^*}\| \|f|F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q_1}\|,\end{aligned}$$

that is, (58) implies (57); here we used Hölder's inequality for $q_1 > q_2$ and the embedding $\ell_{q_1} \hookrightarrow \ell_{q_2}$ when $q_1 \leqslant q_2$. We are left to prove that (57) leads to (58) for $q_1 > q_2$. Here we adapt an example presented by Sickel and Triebel in [25, Section 5.1]. Let $\psi \in \mathcal{S}$ with

$$\operatorname{supp} \mathcal{F}\psi \subset \left\{ \xi\colon\ \xi_1 \leqslant 0,\ \frac{7}{8} N_{l_0} \leqslant |\xi| \leqslant N_{l_0} \right\},$$

recall our notation (11). Let $a_j \in \mathbb{C}$ and $e = (1, 0, \dots, 0)$ be given and put

$$f(x) = \sum_{j=l_0}^{\infty} a_j e^{i\lambda_j\langle e,x\rangle} \psi(x). \tag{79}$$

We use a special partition of unity subordinate to $\Omega^{\boldsymbol{N}}$ (in the sense of Definition 2.4) based on one monotonically decreasing function $\varrho \in C_0^\infty(\mathbb{R})$ with

$$\varrho(t) = 1, \quad |t| \leqslant 1, \qquad \varrho(t) = 0, \quad |t| > \frac{5}{4}.$$

Then $(\varphi_j^{\boldsymbol{N}})_{j\in\mathbb{N}_0}$, given by

$$\varphi_j^{\boldsymbol{N}}(\xi) = \begin{cases} \varrho(N_j^{-1}|\xi|), & j = 0, \dots, l_0 - 1, \\ \varrho(N_j^{-1}|\xi|) - \varrho(N_{j-l_0}^{-1}|\xi|), & j \geqslant l_0, \end{cases} \tag{80}$$

for $\xi \in \mathbb{R}^n$, is a partition of unity in the sense of Definition 2.4, see also [13, Example 2.3.1]. We choose $\lambda_j = N_j - N_{l_0}$, $j = l_0, \dots$, then in a similar way to [25, Section 5.1] we obtain

$$\mathcal{F}^{-1}\left[\varphi_j^{\boldsymbol{N}} \mathcal{F} f\right](x) \sim a_j e^{i\lambda_j\langle e,x\rangle} \psi(x). \tag{81}$$

Consequently, Definition 2.5(ii) leads to

$$\|f|F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q_1}\| \sim \left(\sum_{j=l_0}^{\infty} \sigma_j^{q_1} |a_j|^{q_1} \right)^{1/q_1}, \qquad \|f|F^{\boldsymbol{\tau},\boldsymbol{N}}_{p,q_2}\| \sim \left(\sum_{j=l_0}^{\infty} \tau_j^{q_2} |a_j|^{q_2} \right)^{1/q_2}.$$

But now we are in the same situation as in Step 3, since (57) implies for suitably chosen numbers $b_j = |a_j|\sigma_j$, $j = l_0, \ldots,$

$$\begin{aligned}\left(\sum_{j=l_0}^{\infty} b_j^{q_2}\left(\frac{\tau_j}{\sigma_j}\right)^{q_2}\right)^{1/q_2} &= \left(\sum_{j=l_0}^{\infty} |a_j|^{q_2}\tau_j^{q_2}\right)^{1/q_2}\\ &\leqslant c\left(\sum_{j=l_0}^{\infty} |a_j|^{q_1}\sigma_j^{q_1}\right)^{1/q_1}\\ &= c\|\boldsymbol{b}|\ell_{q_1}\|\end{aligned}$$

(usual modification if $q_1 = \infty$), that is, (68) with

$$r = \frac{q_1}{q_2} > 1, \qquad \alpha_k = \left(\frac{\tau_k}{\sigma_k}\right)^{q_2}, \qquad \beta_k = b_k^{q_2}, \quad k \in \mathbb{N}.$$

Repeating the argument of Step 3 gives (58) and finishes the whole proof. □

Remark 5.4. We briefly return to our examples mentioned in Remark 2.7. Let $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$, and $\boldsymbol{\sigma} = (2^{s_1 j}\Psi_1(2^{-j}))_{j\in\mathbb{N}_0}$, $\boldsymbol{\tau} = (2^{s_2 j}\Psi_2(2^{-j}))_{j\in\mathbb{N}_0}$, where Ψ_1, Ψ_2 are admissible functions, including the 'classical' situation $\Psi_1 \equiv \Psi_2 \equiv 1$, in particular. Then (50) corresponds to the limiting situation

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2},$$

$0 < p_1 \leqslant p_2 \leqslant \infty$. Furthermore,

$$\sigma_j^{-1}\tau_j N_j^{n(\frac{1}{p_1}-\frac{1}{p_2})} = \frac{\Psi_2(2^{-j})}{\Psi_1(2^{-j})}, \quad j \in \mathbb{N}_0,$$

and Theorem 5.3 coincides with [6, Proposition 4.3] in this special setting. In the 'classical' case, i.e., with $\Psi_1 \equiv \Psi_2 \equiv 1$, (54) reduces to $q^* = \infty$, i.e., $q_1 \leqslant q_2$; this coincides with [26, Theorem 2.7.1] and [25, Theorem 3.2.1]. Lizorkin proved part (i) for $p_2 \geqslant p_1 \geqslant 1$ in [20, Theorem D.4.1.7].

In case of the F-spaces the assumption $p_1 < p_2$ in (ii) is essentially used in our argument. Unlike in the B-case there is no interplay between the sequences $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, and $\boldsymbol{N}$ and the fine indices q_i, $i = 1, 2$; for $\Psi_1 \equiv \Psi_2 \equiv 1$ this was already known [26, Theorem 2.7.1(ii)]. The situation changes essentially when approaching the case $p_1 = p_2$ in (iii); this was already studied in [6, Proposition 4.3(iii)] in the context of spaces $F_{p,q}^{(s,\Psi)}$.
Concerning the additional assumption (51), it obviously disappears whenever we are in 'classical' situations, that is, when the upper and lower Boyd indices of the admissible sequences coincide. Then (50) implies that (51) is always true with the difference in (51) being zero. However, in view of the parallel results [7, Lemma 4.1] and [5, Theorem 3.7], see Proposition 3.1 and Remark 3.2, the embedding result may be true in more general situations. But this will be proved elsewhere.

Corollary 5.5. *Let $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < q_1, q_2 \leqslant \infty$, and $\boldsymbol{\sigma}, \boldsymbol{\tau}$ be admissible sequences, $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ admissible with $\underline{N}_1 > 1$. Let q^* be given by* (54). *We assume that*

$$\underline{s}\big(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}\big) \geqslant \bar{s}\big(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2}\big), \tag{82}$$

with the additional assumption (51) *in case of equality in* (82).

(i) *Then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2} \tag{83}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p_2})} \in \ell_{q^*}. \tag{84}$$

(ii) *Let $p_1 < p_2 < \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \hookrightarrow F^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2} \tag{85}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p_2})} \in \ell_{\infty}. \tag{86}$$

(iii) *Let $0 < p_1 = p_2 = p < \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q_1} \hookrightarrow F^{\boldsymbol{\tau},\boldsymbol{N}}_{p,q_2} \tag{87}$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau} \in \ell_{q^*}. \tag{88}$$

Proof. Theorem 5.3 covers the case of equality in (82), so we are left to show the result for $\underline{s}(\boldsymbol{\sigma}\boldsymbol{N}^{-n/p_1}) > \bar{s}(\boldsymbol{\tau}\boldsymbol{N}^{-n/p_2})$. As mentioned before, this implies $\underline{s}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1}\boldsymbol{N}^{-n/p_1+n/p_2}) > 0$, such that $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n/p_1-n/p_2} \in \ell_v$ for any $v \in (0,\infty]$, see Remark 2.1. As for the sufficiency of conditions (84), (86), (88) for the corresponding embeddings, this follows from [5, Theorem 3.7], see also Remark 3.2, for the B-case, and from our above proof for the F-spaces. □

Remark 5.6. To finish we would like to point out that our envelope results can also be applied to the study of approximation numbers. This is based on the inequality:

$$a_{k+1}\big(\mathrm{id}: X(U) \to C(U)\big) \leqslant c k^{-\frac{1}{n}} \mathcal{E}^X_{\mathsf{C}}\big(k^{-\frac{1}{n}}\big), \quad k \in \mathbb{N},$$

which holds for a Banach space X defined on the unit ball U in $\mathbb{R}^n$ with $X(U) \hookrightarrow C(U)$—cf. [16, Corollary 11.18]. In view of our results obtained in [17] one can even expect that the upper estimates obtained in that way can lead to sharp estimates for the approximation numbers of embeddings like $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U) \hookrightarrow C(U)$ or $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U) \hookrightarrow A^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,\infty}(U)$, where the spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U)$ are defined in the usual way by restriction, see also [19, §7]. But this will be studied elsewhere.

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